FIRM VALUATION UNDER UNCERTAINTY – RISK FACTOR MODELLING IN VALUATION

VALORAÇÃO DE EMPRESAS SOB INCERTEZA – MODELAGEM DE FATORES DE RISCO NA AVALIAÇÃO

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ABSTRACT

This paper presents a proposal for business valuation that incorporates business uncertainty conditions inherent to business, which can be represented by continuous-time functions. This approach is yet incipient, although many models are available for research and application in other functions in the discrete field. We incorporate uncertainties external to the company such as prices and resource exhaustion. (ii) Method. We start from Arzac's (2005) proposal – company valuation in continuous time. We include, in the model, business' external uncertainties, such as resources and market price oscillations. (iii) Discussions. In investment decisions, the investor is usually confronted with a set of uncertainties inherent to the business, such as, for example, the oscillatory behavior of revenue, cost competition, etc. In this scenario, the perspective becomes more complex when the business' revenues depend on commodity prices and, additionally, on its resource exhaustion. We incorporate these circumstances that are not captured in traditional models. (iv) Contributions. We argue that, in the neighborhood of the risks internal to the firm and external to the business, investors' decisions must be driven by valuation so that they do not become sunk investments. We propose here a theoretical model that can contribute, in this scenario, to the operators and beneficiaries of the decision-making processes of firm valuation under conditions of uncertainty.

Keywords: Valuation. Cash flow. Uncertainty. Call option.

RESUMO

Este artigo apresenta uma proposta de avaliação de empresas que incorpora as condições de incerteza inerentes ao negócio, as quais podem ser representadas por funções em tempo contínuo. Essa abordagem ainda é incipiente, embora muitos modelos estejam disponíveis para pesquisa e aplicação em outras funções

no campo discreto. Incorporamos incertezas externas à empresa, como preços e exaustão de recursos. (ii) Método. Partimos da proposta de Arzac (2005) – avaliação de empresas em tempo contínuo. Incluímos no modelo as incertezas externas ao negócio, como oscilações de preços de mercado e exaustão de recursos. (iii) Discussões. Nas decisões de investimento, o investidor geralmente se depara com um conjunto de incertezas inerentes ao negócio, como o comportamento oscilatório da receita, concorrência de custos, entre outros. Esse cenário se torna ainda mais complexo quando as receitas do negócio dependem dos preços das commodities e, adicionalmente, da exaustão de seus recursos. Incorporamos essas circunstâncias que não são capturadas pelos modelos tradicionais. (iv) Contribuições. Argumentamos que, diante dos riscos internos à empresa e externos ao negócio, as decisões dos investidores devem ser orientadas por avaliações criteriosas para evitar investimentos irrecuperáveis. Propomos aqui um modelo teórico que pode contribuir, nesse contexto, para operadores e beneficiários dos processos de tomada de decisão na avaliação de empresas sob condições de incerteza.
Palavras-chave: *Avaliação. Fluxo de caixa. Incerteza. Opção de compra*.

1 INTRODUCTION

The instant of entry into a business and the value of the firm are not always the defining elements for the investor's decision whether to invest or not in a firm. Other considerations may come into play when determining his choice of buying or selling stock.

Balance sheets have historical data expressed in the currency in which the transaction was done. According to Martins (Martins, 2001), "input values represent the sacrifice that the company had (past), has (present) or will have (future) to reach a given resource". Yet, according to Martins, the accounting statements that use input values are based on three concepts: profit – as the difference between revenue minus expenses and investment costs –, feasibility, and objectivity. No doubt treating accounting statements as input values is more practical and objective than output values. However, as stated by Martins, historical costs must be avoided in cases where it is intended to verify the future generation capacity of evaluated objective wealth.

According to Iudícibus (2004), accounting data at current prices should be considered when taking decisions because they help recognize unrealized gains or losses. The prices of assets, however, tend to oscillate in different proportions of currency variation while the liabilities are close to current values. On the other side, decision-makers often see themselves powerless to act to soften the "stop and go" movements of unstable economies. Although small inflation rates are not reflected in accounting statements, such factors tend to distort the final result.

Independently of the purchasing power capacity, the context in which the values occurred in the past may not occur in the future. Considering that such values are used as a basis for predictions, we can be led to a naïve assessment of the entity. The assets that generated results in one scenario may not be the same that will generate future results in another one.

When we predict the future based on historical performance and historical growth rates, we may incur imprecise estimates because nothing assures us that such growth will be the same as the past one. The life cycle of assets, its obsolescence, new technologies, new products, new opportunities, the market, and so forth, do not appear in past accounting statements.

Currently, balance sheets are standardized by the rules accorded by the International Financial Reporting Standard (IFRS). One of the most important rules promotes adjustments in the balance sheets to make them closer to the market values. The entities make adjustments in their balance sheets, calling them the "fair value market" of their assets if they are selling or, alternatively, "use value", being chosen the greater one. The use value is given by the present cash flow value projected into the future. It is a simple rule, used to decrease the apparent value of an asset to the company. Thus, if the proposed adjustment is greater than the value in the registers, nothing is changed. Otherwise, the exceeding balance must be lowered. Given this rule, the balances do not fully reflect reality.

Other aspects that can be highlighted are relative to negative performances that may not repeat in the future, from either the market or from the firm. Successful business opportunities, on the other hand, make the firm's future performance positive as opposed to its past.

Therefore, our main contribution is to propose a way to incorporate an additional source of uncertainty in the models, in addition to the uncertainty in business valuation. We use a model derived from the theory of financial options.

1.1 Risks and uncertainties

Following Penrose's definition (2006), uncertainty is linked to the confidence that the entrepreneur has in his estimations and expectations. Risks, on the other hand, are related to the results of his action and its possible related losses.

Risks include both the probability of losses and the amount that is lost. The action of these risks occurs through future projections and estimates made by the exchange of present assets for a future asset.

In the current literature, the principle of increasing risk must be taken into consideration when judging the actions of a company: when increasing its investments, the company also increases its risks with a given probability of loss. Because of this principle, risks act as a limiting factor in the use of capital.

With increasing risks, the entrepreneur must take action to reduce emerging uncertainties and to adjust the company's expansion plan and the company's availability of capital, thus minimizing risks and uncertainties. We note that uncertainties and risks do not always go together; one or the other may occur, or even both together. In a company, it is possible to have a high level of uncertainty with low risk or a high risk with minimal uncertainty. Risks are usually related to the actions to take projects forward. Trust can act as a counterweight of stability against those risks. It makes possible that the available information is able to bring a lower degree of uncertainty. Thus, a company can accept considerable risks in a bet on future status of projects.

According to Penrose (2006), the definition of administrative competence is linked to the skills of the main managers when mobilizing both financial and non-financial resources, such as manpower, etc. The so-called "business sense", which is the manager's sensitiveness to the economic situation, may or may not be present. From the entrepreneurial competence we can refer to the company's ability to make the most of its "energy". By energy we define the firm's capacity of maximizing scarce resources in their purchases, sales, production cycles etc.

The combination of these two traits makes the firm a builder of assets that, if optimized, can lead to the growth of its profits and the expansion desired by its shareholders. Therefore, subjective factors drive the firm's growth by supplanting the objective facts. This, however, does not change the fact that the objective reality is there.

Investments seeking to earn the profits derived from their performance derive from this growth. Investors interested in funding the company's entrepreneurial competence can rapidly increase this growth, depending on their investment rate. Their confidence is based on the belief that their future expectations will be satisfied by regular increases in their business capital. But, as the principle of increasing risk dictates, risk and uncertainty might also grow as well.

Financial reports and statements predict little about future risks, as they do not address managers' actions and how they must act. Cash flows based in estimates might help predicting human-related risks, but they say very little about project risks.

Investors must take advantage of some more relevant information to achieve the desired level of confidence and, by doing this, they are essentially "betting on the future" – they want a return on the benefits that will bring gains over their capital.

Damodaran (2016) defines: "corporate finance as classifying all the decisions made by the company into three major decisions: investment, financing or capital structure and dividends". Thus, the valuation process becomes even more complex when external and internal factors interfere in the valuation process, such as commodity prices and quantities that are running out. In this step, we have adopted a continuous-time model in which we will try to incorporate these uncertainties into a theoretical model.

2 FIRM VALUATION: PRICING METHODS

The first techniques to find stock values date back to the mid-twentieth century, around the 1950s. The beginning of all theoretical development of pricing methods of the shares of a company was the "fundamental principle", developed by Franco Modigliani and Merton Miller (Miller, Modigliani, (1961)) whose study introduced the mechanism for valuing companies under conditions of certainty.

Following this path, Gordon and Shapiro (1956) developed a growth model, where D_t is the dividend rate, g is a growth factor, k is a rate of return demanded by investors in perpetuity:

$$P_0 = \frac{D_1}{(1+k)} + \frac{D_1(1+g)}{(1+k)^2} + \frac{D_1(1+g)^2}{(1+k)^3} + \cdots + \frac{D_1(1+g)^{N-1}}{(1+k)^N} + \cdots$$

Using the formula for the sum of the terms of a geometric progression, we obtain the Gordon growth formula, defined by:

$$P_t = \frac{\frac{D_1}{(1+k)}\left[1-\left(\frac{1+g}{1+k}\right)^N\right]}{1-\frac{1+g}{1+k}}, \qquad P_t = \frac{D_1}{k-g} \tag{1}$$

According to Damodaran (2005), Gordon's model is best suited to companies that are growing at a rate comparable to or lower than the nominal growth rate of the economy, that have well-established dividend payout policies in relation to profits and that intend to continue implementing them in the future. The dividend payout in relation to the company's profits must be consistent with the stability hypothesis, as stable companies generally pay substantial dividends. As a result, the two-stage dividend model emerged.

This model allows for the existence of two stages of growth: an initial phase of high growth and a later equilibrium phase with a stable growth rate, which is expected to remain for the long term.

Assuming that the high-growth phase lasts *n* years, with *g* for percentage growth, we have that:

$$P_0 = \sum_{t=1}^{n} \frac{D_t}{(1+k)^t} + \frac{P_n}{(1+k)^n}, \text{ onde } P_n = \frac{D_{n+1}}{k_n - g_n}, \tag{2}$$

where

- D_t Dividends paid per share for the year t;
- k Expected rate of return (cost of equity) in the high-growth period;
- P_n Price at the end of the year *n*;
- g Extraordinary growth rate for the first *n* years;
- g_n *Perpetual growth rate after one year n*;
- k_n Required rate of return at equilibrium.

Damodaram (2005) developed a model for evaluating growth called (H). This model also presents two stages for the growth of a company, with the difference that in the first stage the growth rate is not constant but decreases linearly over time until it reaches the stable growth rate of the second stage.

The model is based on the assumption that the growth rate of profits begins at a high initial rate g_a and declines linearly over the period of extraordinary growth - amount 2H in duration - to a stable growth rate g_n. It is also assumed that the dividend payout is constant over time, unaffected by changing growth rates.

The value of dividends in the H model can be expressed as:

$$P_0 = \underbrace{\frac{D_0(1+g_n)}{k-g_n}}_{\text{stable growth}} + \underbrace{\frac{D_0 \times H(g_a-g_n)}{k-g_n}}_{\text{extraordinary growth}} , \tag{3}$$

From this model emerged the three-stage growth model, which combines the two-stage model with model H. This model considers the existence of an initial period of high growth, a transition period in which growth declines linearly and a final perpetual phase of stable growth. Furthermore, it does not impose a restriction on the payout rate.

The value of the share in this model is expressed as follows:

$$P_0 = \underbrace{\sum_{t=1}^{t=n_1} \frac{E_0(1+g_a)^t \times \Pi_a}{(1+k)^t}}_{\text{high growth}} + \underbrace{\sum_{t=n_1+1}^{t=n_2} \frac{D_t}{(1+k)^t}}_{\text{transition}} + \underbrace{\frac{E_2(1+g_n) \times \Pi_n}{(k_n-g_n)(1+k_n)^n}}_{\text{stable growth}} , \tag{4}$$

where

- E_t *Profit per share for the year t*;
- D_t *Dividend per share for the year t*;
- g_a Growth rate in the high-growth phase (during n_1 periods);
- g_n Growth rate in the stable growth phase;
- Π_a Payout ratio in the high growth phase;
- Π_n Payout ratio in the stable growth phase;
- k Required rate of return on equity in the period of high growth;
- k_n Required rate of return on equity in the period of stable growth.

The payout rate will generally be lower in the period of high growth, higher during the transition period and higher in the period of stable growth.

That said, we should mention that other models could also be quoted, such as Dechow, Hutton and Sloan (1999), Copeland and Weston (2014), Ohlson (1995), Kenton (2021) and many others. However, our aim at this point is not to give a detailed description of all the models, but rather to try to work through a comparative demonstration of the static model in relation to the dynamic one. To this end, we will address an interesting development rooted in the line of Franco Modigliani and Merton Miller (1961) described above, namely the demonstration of Mao (1969).

2.1 Share pricing and the fundamental principle

In order to demonstrate the valuation aspects of a company, it will be necessary to understand that the revenues from the products marketed less their costs will result in profits. These profits or dividends can be distributed to its shareholders. It should be noted that the assets of this company (by hypothesis) will be valued on the capital market.

The fact is that the components of revenue are linked to variations in prices and quantities inherent in fluctuations in the product market. It should also be stressed that these factors are endogenous to the company, which means that profit or dividends are influenced by factors that are not always controlled by the company and are therefore not modelled under conditions of uncertainty.

In general, the models linked to the Modigliani and Miller principles do not cover in detail the expansion of revenue or costs and, consequently, profits. For a clearer and more exemplary demonstration of the model of development of the conditions of certainty, we have opted to adopt the following notation, based on Mao (Mao, 1969).

P_t	Price of the share at the beginning of year *t*;
n_t	Number of shares available at the beginning of year *t*;
$\Delta n_t = n_{t+1} - n_t$	Change in the number of shares available during year *t*, all new acquisitions (or retentions) are assumed to have occurred at the end of year *t*, at a price P_{t+1}.
$V_t = N_t P_t$	Total value of available shares at the beginning of year *t*;
D_t	Dividends paid per share for the year t, payable at the end of the year;
$D'_t = N_t D_t$	Total of dividends to year *t*, payable at the end of the year;
k	Rate of return required by investors, or cost of equity for the firm
I'_t	Variation in the total amount of resources during year *t*, or net investment during year *t*; it is assumed that all changes occur at the end of the year *t*;
E'_t	Net profit during year *t*, assuming that all profits are received at the end of the year *t*;
$I_t = I'_t / N_t$	Net investment per share along year *t*.

In order to establish the fundamental principle, it is also necessary to assume the following hypotheses:

1. *All investors have a rational behavior and are able to predict the present value and future profitability of any corporation* – therefore, the distinction between shares and debt bonds disappears and all the applications with zero risk produce the same rate of return over the investment.
2. *Investors have a rational attitude concerning any addition to their wealth; it does not matter if this gain is due to dividends or capital gains.*
3. *The capital market is considered to be free of imperfections* – transaction costs do not exist, which means that the asset transference happens with zero risk and there are no differential rates in the taxation of dividends and capital gains.

From these hypotheses, we find out that the returns of different stocks will tend to level out (market clearing). So, the value of the stock would be governed by the stock prices adjusting themselves to a uniform return. Otherwise, investors would just engage in arbitrage, thinking to be more profitable changing their low-return investments to higher-return stocks. So, when the market equilibrium is achieved for any stock, the following fundamental principle must prevail:

$$P_t = \frac{1}{(1+k)}(D_t + P_{t+1}) , \qquad (5)$$

where

$P_t =$	Price of the stock at the beginning of period *t*;
$P_{t+1} =$	Price of the stock at the beginning of period $t+1$;

$D_t =$ Dividend paid per share at the end of period *t;*
$k =$ Return rate of all stocks.

The equation above establishes that, under equilibrium conditions, the sum of paid dividends plus the appreciation of the capital will give a return k on the initial investment, which be equal to the market price.

This principle of determination of value can also applied to the total market value of firms with similar capital. If there are no new stocks issued nor a withdrawal of old ones from the market during the period *t*, one can say that

$$V_t = \frac{1}{(1+k)}(D'_t + V_{t+1}), \tag{6}$$

where

$V_t =$ Market total value of the stocks in the beginning of year *t*;
$V_{t+1} =$ Market total value of the stocks in the beginning of year $t+1$;
$D'_t =$ Total of dividends paid at the end of year *t;*
$k =$ Constant return rate of all stocks.

If during the year *t* the market total value of the stocks increases by issuing new stocks, equation (2) changes to

$$V_t = \frac{1}{(1+k)}(D'_t + V_{t+1} - \Delta n_t P_{t+1}) \tag{7}$$

Where Δn_t is the number of new stocks issued at the end of year *t* at the price P_{t+1}. Additionally $V_{t+1} - \Delta n_t P_{t+1}$ represents the market value of the stocks at the end of year *t*, from the available stocks at the beginning of year *t*.

So, equation (3) shows that under equilibrium conditions the combined returns from dividend payments and capital appreciation along year *t* must give a value *k* over the total value of the stocks at the beginning of year *t*. If Δn_t is negative, equation (3) can also represent the cases in which the shares are retained during year *t*.

2.2 Economic interpretation of fundamental principle

The fundamental principle can be applied to the analysis of the relationship between the determination of the value through the fundamental principle and through the profitability approach. Let's assume that, in order to create the predicted earnings by the investor, the firm has to acquire resources periodically through additional funding. The total amount of new investment that the firm realizes along a given period can be equated to the firm's surplus of current exchange in total resources over the amount of its retained earnings (Amaral, 2008).

This way, let I'_t be the exchange in total resources, E'_t the total earnings and D'_t the total dividends, then $I'_t - (E'_t - D'_t) = \Delta n_t P_{t+1}$ (additional funding), Δn_t is the variation in the number of shares available along the year *t* ($\Delta n_t = n_{t+1} - n_t$). Substituting into (3), we have

$$V_t = \frac{1}{(1+k)}(E'_t - I'_t + V_{t+1}), \tag{8}$$

which is a restatement of the fundamental principle in terms of interest values, investment, and market return rate.

If $t = 1$, equation (4) becomes

$$V_1 = \frac{1}{(1+k)}(E_1' - I_1') + \frac{1}{(1+k)}V_2. \quad (9)$$

Writing V_2 as a function of V_3, V_3 as a function of V_4 and so on, we have

$$V_1 = \sum_{t=1}^{n-1} \frac{1}{(1+k)^t}(E_t' - I_t') + \frac{1}{(1+k)^{n-1}}V_n \,. \quad (10)$$

As n tends to infinity, $\frac{1}{(1+k)^{n-1}}$ tends to zero and the last term disappears. So, equation (10) becomes

$$V_1 = \sum_{t=1}^{\infty} \frac{1}{(1+k)^t}(E_t' - I_t') \quad (k > 0). \quad (11)$$

Equation (7) provides the present value of the shares of the firm, obtained from the fundamental principle. According to this equation, the market value of the shares of a company cannot be understood solely as the sum of future discounted earnings. If additional investments are required in order to create the projected earnings, these disbursements must be compared to the current earnings in those years where the investments were done. These future adjusted earnings, when discounted by the present value, provide the market value of the company's shares. So, equation (11) is consistent with the approach by profits method to find the values of shares.

2.3 Alternative interpretations of the fundamental principle

Due to the hypotheses above, equation (11) can be interpreted according to three other methods: by the opportunity of investments, by the dividends and by the discounted cash flow.

2.3.1 Opportunity of investments approach

According to this approach, the market value of the firm’s shares is equivalent to the present value of existing resources plus the future value of opportunities of investments. The value of existing resources is equal to the capitalized value of net profits generated by existing resources. The future value of investment opportunities is equal to the present value of future investments multiplied by a rate of return for these investments. We show below that the value of shares obtained this way is identical to that obtained according to equation (11).

2.3.2 Discounted cash flows

In this approach the market value of a company is equal to the present value of all cash flows between shareholders and the company plus the value of future investment opportunities. The business is considered like a machine that allows the owners to make periodic cash deposits or withdrawals. The payment of the shareholders is treated both as dividends paid or as shares withdrawn. The shareholders contribute to the company by purchasing new shares. As previously shown, $\Delta n_t P_{t+1}$ (amount of additional funding of the company) is equal to $I_t' - (E_t' - D_t')$, that is, the excess of net investment over retained earnings.

From the previous expression, it is easy to deduce that $E_t' - I_t' = D_t' - \Delta n_t P_{t+1}$. This last equation shows that $E_t' - I_t'$ is identical to the cash flow between shareholders and the company during year t. Thus, equation (7) is consistent with the discounted cash flow approach.

2.3.3 Dividend payment

The market value of a company, under the dividend payment approach is equal to the present value of all future dividend payments discounted by the market rate of return. The difference between this approach and the discounted cash flow approach lies in the relation 〖E_t^'-I〗_t^'=D_t^'-Δn_t P_(t+1) . This shows that the company's cash flow to the stockholders is greater than the dividends paid in the same period when old zero-risk investments are withdrawn and lesser when new zero-risk investments are sold. If the earnings of zero-risk investment are always zero, the cash flow is always equal to the dividend payment and the two approaches become clearly identical. It can also be shown that the dividend payment approach is consistent with the determination of values in equation (11) and, thus, equivalent to the discounted cash flow approach even when the earnings of the risk-zero investment are not zero (Mao, 1969). In the following, we present valuation models in line with the fundamental principle and these interpretations.

2.4 Mathematical equivalence between other valuation theories

If we assume certainty, rationality and imperfection-free capital markets, the four evaluation models are mathematically equivalent. This proof, given by Modigliani and Miller, which is reproduced here with few modifications, establishes this equivalence by showing that the four models are formally equivalent to the fundamental principle of evaluation described by equation (11). Thus, if the four approaches are equivalent to the fundamental principle, they must be equivalent with each other. Considering this, there are four other approaches that can be derived.

2.4.1 Investment opportunities approach

This approach determines the value of the shares of a company through the sum of the present value of existing resources with the present value of future investment opportunities. Thus, if the existing resources of a company generate constant annual earnings E_1' and if the company has the opportunity to reinvest a fraction b of each annual earning at a return r per year, the current market value of shares V_1 is given by

$$V_1 = \frac{E_1'}{k} + \sum_{t=1}^{\infty} \left[\frac{bE_1'(1+br)^{t-1}}{(1-k)^t}\right]\left(\frac{r-k}{k}\right) (k > 0). \quad (12)$$

Observe that expression $bE_1'(1+br)^{t-1}$ expresses the amount of reinvestment in the year t (t = 1, 2,…). If $bE_1'(1+br)^{t-1}$ is designated simply as I_t', equation (12) becomes

$$V_1 = \frac{E_1'}{k} + \sum_{t=1}^{\infty} \frac{I_t'}{(1+k)^t}\left(\frac{r-k}{k}\right), \quad (13)$$

where all terms are defined as above.

To show that equation (13) is implicitly linked to the fundamental principle, Modigliani and Miller noted that E_t', the amount of earnings in year t, is equal to E_1', the constant annual earnings obtained from the existing resources, plus realized investment returns from year 1 to the year $t-1$. That is:

$$E_t' = E_1' + r(I_1' + I_2' + \cdots + I_{t-1}').$$

Substituting this in (7), the fundamental principle can be formulated as

$$\begin{aligned} V_1 &= \sum_{t=1}^{\infty} \frac{1}{(1+k)^t}(E_t' - I_t') \\ &= \frac{1}{(1+k)}(E_t' - I_t') + \sum_{t=2}^{\infty} \frac{1}{(1+k)^t}[E_1' + r(I_1' + I_2' + \cdots + I_{t-1}') - I_t'] \\ &= \sum_{t=1}^{\infty} \frac{E_1'}{(1+k)^t} + \sum_{t=2}^{\infty} \frac{r}{(1+k)^t}[I_1' + I_2' + \cdots + I_{t-1}'] - \sum_{t=1}^{\infty} \frac{I_t'}{(1+k)^t}. \end{aligned} \tag{14}$$

Note that $\sum_{t=1}^{\infty} \frac{E_1'}{(1+k)^t}$ is equal to $\frac{E_1'}{k}$ and that $\sum_{t=2}^{\infty} \frac{r}{(1+k)^t}[I_1' + I_2' + \cdots + I_{t-1}']$ is equal to $\sum_{t=1}^{\infty} I_t' \left(\sum_{\tau=t+1}^{\infty} \frac{r}{(1+k)^\tau}\right)$, that is also equal to $\sum_{t=1}^{\infty} I_t' \left(\frac{r}{k(1+k)^t}\right)$.

Thus, equation (14) can be simplified to

$$V_1 = \frac{E_1'}{k} + \sum_{t=1}^{\infty} I_t' \frac{r}{k(1+k)^t} - \sum_{t=1}^{\infty} \frac{I_t'}{(1+k)^t}. \tag{15}$$

Equation (15), after some simplifications, becomes identical to (13), and represents the opportunity of investments approach:

$$V_1 = \frac{E_1'}{k} + \sum_{i=1}^{\infty} \frac{I_t'}{(1+k)^t}\left(\frac{r-k}{k}\right). \tag{16}$$

This proves that the opportunity of investments approach and the fundamental principle of evaluation lead the shares of a company to the same value, if the investment is converted into a contribution of capital and the expected return is the same.

The last two equations can be written as:

$$\begin{aligned} &\sum_{t=2}^{\infty} \frac{r}{(1+k)^t}[I_1' + I_2' + \cdots + I_{t-1}'] \\ &= \left[I_1' \sum_{t=2}^{\infty} \frac{r}{(1+k)^t} + I_2' \sum_{t=3}^{\infty} \frac{r}{(1+k)^t} + I_3' \sum_{t=4}^{\infty} \frac{r}{(1+k)^t} + \cdots\right]. \\ &= \sum_{t=1}^{\infty} I_t' \left[\sum_{\tau=t+1}^{\infty} \frac{r}{(1+k)^t}\right] \end{aligned} \tag{17}$$

Furthermore,

$$\sum_{\tau=t+1}^{\infty} \frac{1}{(1+k)^\tau} = \sum_{t=1}^{\infty} \frac{1}{(1+k)^t} - \sum_{\tau=1}^{t} \frac{1}{(1+k)^\tau} = \frac{1}{k(1+k)^t}. \tag{18}$$

So,

$$\sum_{t=1}^{\infty} I_t' \left[\sum_{\tau=t+1}^{\infty} \frac{r}{(1+k)^t}\right] = \sum_{t=1}^{\infty} I_t' \frac{r}{k(1+k)^t}. \tag{19}$$

2.4.2 Mathematical equivalence to profits approach

The profits approach equates the value of a company's share to the present value of all adjusted future profits. If a company has to make additional investments to generate the projected profits, the cost of these investments must be deducted from current profits in the years these investments are made. The present value of these adjusted profits determines the value of a company's share market.

As an example, suppose a certain company expects to generate profits $E_1', E_2', \ldots$, in years 1, 2,..., and so on. To generate this series of future profits, the company must invest $I_1', I_2', \ldots$, in years 1, 2,..., and so on. Adjusted profits will be equal to $E_1' - I_1'$ in year 1, $E_2' - I_2'$ in year 2 and, generically, $E_t' - I_t'$ in year *t*. So, the market value of the shares, V_1, is given by

$$V_1 = \sum_{t=1}^{\infty} \frac{E_t' - I_t'}{(1+k)^t}. \tag{20}$$

But this expression is exactly the same equation that expresses the fundamental principle of stock valuation. Thus, the profits approach is consistent with the fundamental principle.

2.4.3 Dividends payment approach

The dividend approach equates the value of a company's share to the present value of all future dividends paid on the share. Suppose that $D_{t,1}'$ denotes the total amount of dividends paid in year *t* on shares starting in year 1. If dividend payments are expected for the company's shares $D_{1,1}', D_{2,1}', \ldots$ etc., the current market value of shares, V_1, is given by

$$V_1 = \sum_{t=1}^{\infty} \frac{D_{t,1}'}{(1+k)^t}. \tag{21}$$

Equation (17) is also equivalent to the fundamental principle described by equation (7). If new shares are not sold after year *1*, then $D_{t,1}'$, the dividends paid in year *t* on the shares starting in year 1, equals D_t', the total sum of dividends paid in year *t*. Furthermore, provided there is no external funding, the dividend payment D_t' is equal to profit surplus over investments, that is, $E_t' - I_t'$. Replacing $E_t' - I_t'$ by $D_{t,1}'$ we see that equation (17) becomes identical to equation (7).

Next, suppose that new shares were sold after year 1. To show that equations (17) and (7) are still equivalent, Modigliani and Miller rewrote equation (12) as

$$\begin{aligned} V_1 &= \sum_{t=1}^{\infty} \frac{D_{t,1}'}{(1+k)^t} \\ &= \frac{D_{1,1}'}{(1+k)} + \sum_{t=2}^{\infty} \frac{D_{t,1}'}{(1+k)^t} \\ &= \frac{1}{(1+k)}\left[D_{1,1}' + \sum_{t=1}^{\infty} \frac{D_{t+1,1}'}{(1+k)^t}\right] \end{aligned} \tag{22}$$

We can see that the sum of dividends paid, starting in year 2, and those resulting from shares starting in year 1, is equal to the result of dividends paid on all the shares starting in year 2, minus the results of dividends of new shares paid during the year 1. Thus,

$$\sum_{t=1}^{\infty} \frac{D_{t+1,1}'}{(1+k)^t} = \sum_{t=1}^{\infty} \frac{D_{t+1,2}'}{(1+k)^t}\left(1 - \frac{\Delta n_1}{n_2}\right). \tag{23}$$

This way, equation (18) can be written as

$$V_1 = \frac{1}{(1+k)}\left[D'_{1,1} + \sum_{t=1}^{\infty} \frac{D'_{t+1,2}}{(1+k)^t}\left(1 - \frac{\Delta n_1 P_2}{n_2 P_2}\right)\right]. \tag{24}$$

Both expressions, $\sum_{t=1}^{\infty} \frac{D'_{t+1,2}}{(1+k)^t}$ and n_2P_2, in equation (24) represent V_2, the value of shares of the company in the beginning of year 2. Even more, the obligation of all dividends paid in any period results in shares starting in that period: $D'_{1,1} = D'_1$. Equation (24) can be simplified to

$$V_1 = \frac{1}{(1+k)}(D'_1 + V_2 - \Delta n_1 P_2). \tag{25}$$

Since equation (25) is consistent with equation (11), which was shown in order to arrive at equation (11), equation (25) is also consistent with equation (11).[6]

Among the vast literature on company valuation, the discounted cash flow method stands out as the most used by financial analysts. Over time, the need has emerged to incorporate uncertainty factors into the equations (Bottazzi, 2023).

2.4.4 Discounted cash flow approach- modelling in continuous time

The mainstream approach equates the market value of the company's shares to the present value of all future cash flows between rights holders and the company. This approach differs from the dividend payment approach in which the cash flow includes the net variation of available shares as well as the cash flow for paying dividends. It was shown earlier that the cash flow in any period is equal to E_t^'-I_t^', the difference between earnings and investments for that period. So, the discounted cash flow approach is also consistent with the fundamental principle in equation (11).

In this sense, following Brennan and Schwartz's (1984) explanation, the model partly neglects the valuation of companies. That is because, in its primary form, the model seeks equilibrium through permanent rates of return, without considering their variability. Thus, capital rates change with subsequent capital, i.e. the model is fixed on endogenous factors. Furthermore, it fails to adequately capture changes in the preferences of individual investors over a period of time. Also according to the authors, the only asset pricing model with average variance that, until then, tried to express equilibrium prices according to the characteristics of future cash flows, without making unjustified assumptions about endogenous facts, was that of Stapleton and Subrahmanyam (1978), which, however, resulted in calculations that were not very applicable.

In this context, Brennan and Schwartz (1984) state that a more complete model should include some aspects such as the fact that risk is associated with multiple periods; quantifiable variables with clear empirical counterparts; companies' investment alternatives; different investment opportunities and others. These authors worked with differential equations that contained stochastic movements for profits and investment policies. This model was based on theories of the evolution of securities priced on the market, which evolved into other models, including Arzac's (2005) continuous time model, which we will use to develop a set of two simultaneous uncertainties, price and quantity.

Before that, however, we'd like to give a short introduction to continuous-time models.

[6] For a complete exploration, see Mao (1969).

3 MODELLING UNCERTAINTIES – THE BROWNIAN MOTION

Uncertainties about the business can imply issues with the company value. The revenue performance is uncertain in most cases. Still, it is possible to model the evolution of the revenue in time considering uncertainties. In this article, we use the geometric Brownian movement in continuous time to show this.

In 1827 the Scottish botanist Robert Brown made an important observation about the movement of pollen organelles suspended in water. The trajectories of these particles under a microscope appeared random and, in a time when the atomic constitution of matter was yet to be established, scientists could only speculate why. Later, in 1905, Albert Einstein gave a physical explanation for these observations and characterized the Brownian motion (BM) as it was called, as collisions between water molecules and the suspended particles, confirming the atomic constitution of matter. The mathematical theory of the BM was established in 1921 by Nobert Wiener, the most important example of stochastic process in continuous time t. It was also called the Wiener process, denoted by $W(t)$ in more than one dimension.

The basic properties of the Brownian motion assume that $W(t)$ is a collection of random variables for each $t \geq 0$ defined on a sample space with probability P, satisfying (Campolieti & Makarov, p. 375)

(i) Initial condition: $W(0) = 0$.
(ii) Independent increments: if $0 \leq t_0 < t_1 < \cdots < t_n$, then $W(t_i) - W(t_{i-1}),\ 1 \leq i \leq n$, are jointly independent random variables.
(iii) Normal distribution: If $0 < s < t$ then $W(t) - W(s) \sim N(0, t - s)$ in P.
(iv) The trajectories are continuous functions of t (in the probabilistic sense).

The use of this motion to model uncertainty in the financial market stems from the fact that the log of the price is assumed to be normally distributed, and the price variations are independent. This is similar to the historical motion of pollen organelles from collisions of water molecules where the displacement of an actual collision is independent of past displacements. All of this is a consequence of property (ii).

4 ITÔ’S LEMMA

An important development in the theory of stochastic processes is the solution of differential equations involving the Wiener process. A rigorous theory is formulated in integrals, notably the Itô’s integral (Itô, 2004). For applied purposes it is mostly expressed in terms of abstract differentials that follow practical rules. In this paper we express the revenue of the firm R as a percentual variation which, in increment form, is given by $\Delta R/R = \alpha \Delta t + \beta \Delta W$. In other words, the percent variation is an arithmetic Brownian motion with coefficients $\alpha \in \mathbb{R}$ and $\beta > 0$ whose interpretation will be given in terms of the firm revenue.

We are interested in obtaining the solution of the equation where the infinitesimal percentual increment follows a Brownian motion with constant coefficients,

$$dY = \alpha Y dt + \beta Y dW \qquad (26)$$

where $dW \sim N(0, dt)$. This is a simple example of a large class of equations known in the literature as “stochastic differential equations” (Mikosch, 2000; Malliaris, 1990). We demonstrate the solution below.

Suppose we have a differentiable two variable function, $Y = f(t, W)$, and f expands as a Taylor series in both t and W. Using the well-known stochastic calculus heuristics $dt^2 \leftrightarrow 0$, $dtdW \leftrightarrow 0$, $dW^2 \leftrightarrow dt$ we arrive at one of the Itô's equations (the only one needed here):

$$df(t,W) = \left(f_1(t,W) + \tfrac{1}{2} f_{22}(t,W)\right) dt + f_2(t,W)dW \tag{27}$$

where the indexes refer to the order of derivatives with respect to the first and second variables, respectively. Regarding this formula, it is evident that it is not necessary differentiate of all orders, but it is sufficient that $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$. Using this formula, it is immediate to show that the expression

$$f(t,W) = f_0 \exp\left\{(\alpha - \tfrac{1}{2}\beta^2)t + \beta W\right\} \tag{28}$$

is a solution to the stochastic differential equation $dY = \alpha Y dt + \beta Y dW$, which is also called geometric Brownian motion (GBM) with $f(0, W(0)) = f_0$, constant.

5 CONTINUOUS-TIME FREE CASH FLOW MODEL

Some evaluation models use discrete periods where accounting considerations are applied at the end of each period. Discrete processes may represent the behaviour of a temporal series in a simpler way, but it is considered better to employ a continuous model where there is a description that captures the essential features of the dynamics. In our case, the revenue (and the free cash flow) can be represented by the GMB discussed briefly.

According to Arzac (Arzac, 2005), using continuous time has many advantages:

1 – It allows compacting a transparent model of arbitrage price equation associated with inputs and outputs options of the firm.

2 – It can capture the complexity of decisions and incorporate relevant information: total cost or entry cost at different points in time, temporal evolution of free cash flows as a function of revenues and volatilities and also exit costs or continuation.

3 – Some companies' entry and exit options do not have a defined end and cannot be represented in a decision tree (binomial model that uses discrete time), but they still can be modelled in continuous time.

4 – The general solution of pricing equations is provided by exact forms or through numerical calculations.

We define R as the revenue and Y as the free cash flow of the firm. It is assumed a linear relationship between them, so that the revenue is represented by a GBM (Arzac, 2005),

$$Y = \mathrm{b}R$$
$$\frac{\mathrm{d}R}{R} = \alpha \mathrm{dt} + \beta dW \tag{29}$$

where $b > 0$, α is the expected growth rate for revenue and β the volatility of dR/R.

The GBM represents a reasonable behavior of stock prices and can be applied in this situation. It is used to model the behavior of economic variables that cannot have negative values and that increase with time, but that are subject to uncorrelated random changes with their expected growth rates.

6 ENTERPRISE EVALUATION IN CONTINUOUS TIME

Our intent is to describe the value of an enterprise using the continuous time Brownian motion. It is enough to discount in continuous time the free cash flow also obtained in continuous time. This is feasible through the calculation of the current value using integration. In what follows a and b are constants, τ the tax rate of the company, r_d the indebtedness rate, ω is the weighted average cost of capital and $R(t)$ the GBM that describe the revenue with $E[R(t)] = R_0 e^{\alpha t}$. Then, according to Arzac (Arzac, 2005) the value of the enterprise is

$$
\begin{aligned}
V(R) &= a\int_0^{\infty} exp\,(-(1-\tau)r_d t)dt + b\int_0^{\infty} E(R)exp\,(-\omega t)dt \\
V(R) &= a\left[\frac{exp\,(-(1-\tau)r_d t]_0^{\infty}}{-(1-\tau)r_d} + bR_0\int_0^{\infty} exp\,(-(\omega-\alpha)t)dt\right] \\
V(R) &= \frac{a}{(1-\tau)r_d} + \frac{bR_0}{\omega-\alpha}
\end{aligned}
\tag{30}
$$

where $(1\text{-}\tau)r_d$, ω and α denote the equivalent rates in continuous time if we considered this model in discrete time. This value is a little greater than the value found to cash flow in discrete time because, according to Arzac (Arzac, 2005), the formulation in continuous time allows cash flows to arrive continuously through time and not only at the end of discount period.

7 COMPANY VALUATION UNDER UNCERTAINTY CONDITIONS

We use the case of a company that owns the right to explore an oil well as an example. Besides the uncertainty in revenue due to oil price volatility we also consider an uncertainty in the quantity K of oil available in the well and represent its expected value by $\overline{K}$. We assume that the oil price $p(t)$ follows a GBM given by

$$dp = \alpha p dt + \sigma p dW, \tag{31}$$

Where:

α - expected growth rate of oil price.
σ - oil price volatility.

Profits Y are defined in the following way:

$$Y = bqp + a \tag{32}$$

Where:

b – revenue profit margin (0 < b < 1).
q – constant flow with which the oil is removed from the well (quantity /time).
a – fixed cost.

As the quantity of oil to be extracted is finite, its lifespan T must equal

$$T = \frac{K}{q} \tag{33}$$

with an uncertain quantity K and assuming that oil is extracted at a constant flow rate q.

The question is: what is the value of this company? Since the reservoir is finite, the integration must consider time T as an upper limit in time (Arzac, 2005).

$$V(p,T) = bq\int_0^T E(p)\exp(-\omega t)dt + a\int_0^\tau \exp[-(1-\tau)r_d t]dt \quad (34)$$

Using $E[p(t)] = p_0 e^{\alpha t}$,

$$V(p,T) = \frac{bqp_0}{\omega-\alpha}[1-\exp[-(\omega-\alpha)T]] + \frac{a}{(1-\tau)r_d}[1-\exp[-(1-\tau)r_d T]] \quad (35)$$

We consider the variances in K and T, respectively σ_K^2 and σ_T^2, to be related by

$$\sigma_T^2 = \frac{\sigma_K^2}{q^2} \quad (36)$$

Now we propose that the value of this company be determined by the following formula,

$$V(p) = \int_{-\infty}^{\infty} V(p,T)P(T)dT. \quad (37)$$

Where the probability function *P(T)* is a Gaussian one with average $\bar{T} = \bar{K}/q$ and variance σ_T^2. That is, with a single source of uncertainty,

$$V(p) = \int_{-\infty}^{\infty}\left\{\frac{bqp_0}{\omega-\alpha}[1-\exp[-(\omega-\alpha)T]] + \frac{a}{(1-\tau)r_d}[1-\exp[-(1-\tau)r_d T]]\right\}\exp\left[\frac{-(T-\bar{T})^2}{2\sigma_T^2}\right]\frac{dT}{\sqrt{2\pi\sigma_T^2}} \quad (38)$$

Changing variables, $u = T - \bar{T}$, $du = dT$, and using the well-known expression (Abramowits & Stegun, 1972);

$$\int_{-\infty}^{\infty} du \cdot e^{-mu^2+mu} = e^{\frac{n^2}{4m}}\sqrt{\frac{\pi}{m}}, \text{ with } m \in \mathbb{R}^+. \quad (39)$$

We arrive at

$$V(p) = \frac{bqp_0}{\omega-\alpha}\left\{1-\exp\left[\frac{(\omega-\alpha)^2\sigma_T^2}{2} - (\omega-\alpha)\bar{T}\right]\right\} + \frac{a}{(1-\tau)r_d}\left\{1-\exp\left[\frac{(1-\tau)^2 r_d^2\sigma_T^2}{2} - (1-\tau)r_d\bar{T}\right]\right\} \quad (40)$$

This equation is the integration of $V(p)$ given above. Note that that equation may be applied to any uncertain parameter relevant in the value of an enterprise since its uncertainty can be represented by a probability distribution.

8 A “TOY MODEL” FOR A CALL OPTION OF AN OIL WELL

So far, we considered the impact of uncertainties in the company value. Now we will evaluate how external uncertainties can influence the valuation. We will study the value of a call option of the same oil well previously analysed.

Consider an investor interested in exploiting the oil well and who wants to know the value of a call premium. We consider the amount of oil in the well to be given by a lognormal variable K in barrels, with expected value, or average, $\overline{K}$, and standard deviation σ_K. Let P_t be the price per oil barrel, a geometric Brownian motion, T the lifespan, starting at $t = 0$, and I_x the call price of extracting the average amount of oil from the well.

We assume that the average price of the well is given by

$$V_T = P_T\overline{K} \quad (41)$$

The investment necessary to extract this oil is

$$P_S\overline{K} = I_x \quad (42)$$

where P_S is the per barrel exercise price of the oil. Considering the *payoff* $Max[V_T - I_x, 0]$ the option will be exercised if $P_T > P_S$. From these considerations,

$$C_T(P_S) = \overline{K}.C_1(P_S), \quad (43)$$

where $C_T(P_S)$ is the premium to extract the average amount and $C_1(P_S)$ is the premium to extract an additional barrel.

If there are two sources of uncertainty, we define the *payoff* R_T as

$$R_T = 0 \quad IF\ P_T < P_S;$$
$$R_T = (P_T - P_S)K \quad IF\ P_T > P_S. \quad (44)$$

The real option *payoff* with one source of uncertainty is:

$$\Pi_T = 0 \quad IF\ P_T < P_S;$$
$$\Pi_T = (P_T - P_S)\overline{K} \quad IF\ P_T > P_S. \quad (45)$$

The return difference between these *payoffs* is

$$\Delta = 0 \quad IF\ P_T < P_S;$$
$$\Delta = (K - \overline{K})(P_T - P_S) \quad IF\ P_T > P_S. \quad (46)$$

Assuming independence between P_T and *k*,

$$E[\Delta] = 0 \quad (47)$$

Also, for random independent variables X and Y we have $Var[X\,Y] = Var[X]Var[Y] + Var[Y](E[X])^2 + Var[X](E[Y])^2$. Applying this to $X = K - \overline{K},\ Y = P_T - P_S$, and taking σ_E^2 as the variance of XY, we have the enterprise variance

$$\sigma_E^2 = \alpha\sigma_K^2 \quad (48)$$

where σ_K^2 is the variance of k and α a positive function.

What would be the value of the call option with two sources of uncertainty? If two sources increase the uncertainty concerning one source, the answer should be no less than $\overline{K}C_T(P_S) + \lambda\sigma_E$,where λ is the market price of risk.

9 GENERAL MODEL FOR ENTERPRISE EVALUATION UNDER UNCERTAINTY CONDITIONS

Besides business inherent uncertainties, exogenous company-related external uncertainties incorporated in economic predictions, market risk evaluation, and others, may also contribute to the determination of the enterprise value. We suggest a model that incorporates a second source of uncertainty that increases risk:

$$V_E = V(p) + \lambda\sigma_E \quad (49)$$

Where:

V_E – value of enterprise with two sources of uncertainty according to the discussion above.
V(p) – value of enterprise.
λ – Market risk price.

The greater the risk price given by the market and the business' inherent uncertainty, the greater will be the market discount to evaluate the value of the enterprise. If the inclusion of a second source of uncertainty does not imply additional risk to the business, the value of λ will be zero. Thus, equation (49) is the more general way to incorporate additional uncertainty to business value because it allows both internal and external uncertainties.

In the example of the oil well, λ is a value given by the market and σ_E is a function of both uncertainties, σ_K and σ_P. With two sources of uncertainty that increase risk, we should add $\lambda\sigma_E$ to the value calculated in equation (40) to obtain a possible minimum value of the enterprise.

CONCLUSION

The uncertainties in an enterprise may generate great disturbances in the temporal series and affect its value of the enterprise. In this work we analysed a simple mathematical way to incorporate the effect of a second source of uncertainty in the evaluation of an enterprise and reached a proposal for the minimum acceptable value. In investment decisions, the investor faces a set of uncertainties as, random behavior of revenue, competition from other investors and depletion of natural resources. Decisions must be taken about resources and minimization of risks. Postponing entry may result in losses due to an increase in the number of potential competitors and several options need to be analysed before defining a strategy.

It is important to observe that investment decisions do not only follow mathematical rules. Normally investors have other more significant information to achieve the desired confidence degree and so they "bet" in the business. That is due to the uncertainties about the future that any business intrinsically has. These uncertainties make the application of mathematical techniques a complement to decision-making. If the risks are known, each investor develops his "decision rules" based on his own strategic vision in the approached development.

This paper presented another approach in company evaluation. This approach proposes incorporating uncertainties using a call options model providing an additional analytical instrument that can help the strategic visions of each entrepreneur in making decisions of investments.

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