A note on the continuity of Oseledets subspaces for fiber-bunched cocycles*

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Abstract

We prove that restricted to the subset of fiber-bunched elements of the space of $GL(2,\mathbb{R})$-valued cocycles Oseledets subspaces vary continuously, in measure, with respect to the cocycle.

1 Introduction

In its simple form, a linear cocycle is just an invertible dynamical system $f: M \to M$ and a matrix-valued map $A: M \to GL(d,\mathbb{R})$. Sometimes one calls linear cocycle (over f generated by A), instead, the sequence $\{A^n\}_{n\in\mathbb{Z}}$ defined by

$$A^n(x) = \begin{cases} A(f^{n-1}(x))\dots A(f(x))A(x) & \text{if } n > 0 \\ Id & \text{if } n = 0 \\ A(f^n(x))^{-1}\dots A(f^{-1}(x))^{-1} & \text{if } n < 0 \end{cases}$$

for all $x \in M$.

A special class of cocycles is given when the base dynamics f is hyperbolic and the dynamics induced by A on the projective space is dominated by the dynamics of f. That is, the rates of contraction and expansion of the cocycle A along an orbit are smaller than the rates of contraction and expansion of f. Such a cocycle is called *fiber-bunched* (see Section 2 for the precise definitions).

Many aspects of fiber-bunched cocycles are rather well understood. For instance, it is known that their cohomology classes are completely characterized by the information on periodic points [2, 8], generically they have simple Lyapunov spectrum [5, 9] and in the case when $d = 2$, Lyapunov exponents are continuous as functions of the cocycle [3]. In this short note, still in the context of fiber-bunched cocycles, we address the problem of continuity of the Oseledets subspaces. More precisely, we prove that restricted to the subset of fiber-bunched elements of the space of $GL(2,\mathbb{R})$-valued cocycles Oseledets subspaces vary continuously, in measure, with respect to the cocycle. The proof of this result relies on ideas from [3] and [4]. In a different context a similar statement was recently gotten by [6].

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2 Definitions and statements

Let (M, d) be a compact metric space and $f : M \to M$ be a homeomorphism. Given any $x \in M$ and $\varepsilon > 0$, we define the *local stable* and *unstable sets* of x with respect to f by

$$W^s_\epsilon(x) := \{y \in M : \mathsf{d}(f^n(x), f^n(y)) \leq \epsilon,\ \forall n \geq 0\},$$
$$W^u_\epsilon(x) := \{y \in M : \mathsf{d}(f^n(x), f^n(y)) \leq \epsilon,\ \forall n \leq 0\},$$

respectively.

Following [1], we say that a homeomorphism $f : M \to M$ is *hyperbolic with local product structure* (or just *hyperbolic* for short) whenever there exist constants $C_1, \epsilon, \tau > 0$ and $\lambda \in (0, 1)$ such that the following conditions are satisfied:

- $\mathsf{d}(f^n(y_1), f^n(y_2)) \leq C_1\lambda^n \mathsf{d}(y_1, y_2)$, $\forall x \in M$, $\forall y_1, y_2 \in W^s_\epsilon(x)$, $\forall n \geq 0$;
- $\mathsf{d}(f^{-n}(y_1), f^{-n}(y_2)) \leq C_1\lambda^n \mathsf{d}(y_1, y_2)$, $\forall x \in M$, $\forall y_1, y_2 \in W^u_\epsilon(x)$, $\forall n \geq 0$;
- If $\mathsf{d}(x, y) \leq \tau$, then $W^s_\epsilon(x)$ and $W^u_\epsilon(y)$ intersect in a unique point which is denoted by $[x, y]$ and depends continuously on x and y. This property is called *local product structure.*

Fix such an hyperbolic homeomorphism and let $A : M \to GL(d, \mathbb{R})$ be a r-Hölder continuous map. This means that there exists $C_2 > 0$ such that

$$\|A(x) - A(y)\| \leq C_2 \mathsf{d}(x, y)^r \text{ for any } x, y \in M.$$

Let us denote by $H^r(M)$ the space of such r-Hölder continuous maps. We endow this space with the r-Hölder topology which is generated by norm

$$\| A \|_r := \sup_{x \in M} \| A(x) \| + \sup_{x \neq y} \frac{\| A(x) - A(y) \|}{d(x, y)^r}.$$

We say that the cocycle generated by A satisfies the *fiber bunching condition* or that the cocycle is *fiber-bunched* if there exists $C_3 > 0$ and $\theta < 1$ such that

$$\|A^n(x)\|\|A^n(x)^{-1}\|\lambda^{nr} \leq C_3\theta^n$$

for every $x \in M$ and $n \geq 0$ where λ is the constant given in the definition of hyperbolic homeomorphism.

Let μ be an ergodic f-invariant probability measure on M with local product structure. Roughly speaking, the last property means that μ is locally equivalent to the product measure $\mu^s \times \mu^u$ where μ^s and μ^u are measures on the local stable and unstable manifolds respectively induced by μ via the local product structure of f. Since we are not going to use explicitly this property we just refer to [3] for the precise definition.

It follows from a famous theorem due to Oseledets (see [10]) that for μ-almost every point $x \in M$ there exist numbers $\lambda_1(x) > \ldots > \lambda_k(x)$, and a direct sum decomposition $\mathbb{R}^d = E^{1,A}_x \oplus \ldots \oplus E^{k,A}_x$ into vector subspaces such that

$$A(x)E^{i,A}_x = E^{i,A}_{f(x)} \text{ and } \lambda_i(x) = \lim_{n\to\infty} \frac{1}{n} \log \| A^n(x)v \|$$

for every non-zero $v \in E_x^{i,A}$ and $1 \leq i \leq k$. Moreover, since our measure μ is assumed to be ergodic the *Lyapunov exponents* $\lambda_i(x)$ are constant on a full μ-measure subset of M as well as the dimensions of the *Oseledets subspaces* $E_x^{i,A}$. Thus, we will denote by $\lambda^-(A,\mu) = \lambda_k(x)$ and $\lambda^+(A,\mu) = \lambda_1(x)$ the *extremal Lyapunov exponents* and by $E_x^{s,A} = E_x^{k,A}$ and $E_x^{u,A} = E_x^{1,A}$ the *stable and unstable spaces* respectively. It follows by the Sub-Additive Ergodic Theorem of Kingman (see [7] or [10]) that the extremal Lyapunov exponents are also given by

$$\lambda^+(A,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|$$

and (2.1)

$$\lambda^-(A,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|(A^n(x))^{-1}\|^{-1}$$

for μ almost every point $x \in M$. The objective of this note is to understand, for a fixed base dynamics f, how does the map $A \to E_x^{i,A}$ vary in the case when $d = 2$, that is, in the case when the cocycle A takes values in $GL(2,\mathbb{R})$.

Let d be the distance on the projective space $\mathbb{P}(\mathbb{R}^2)$ defined by the angle between two directions. We say that an element A of $H^r(M)$ with $\lambda^+(A,\mu) > \lambda^-(A,\mu)$ is a *continuity point for the Oseledets decomposition with respect to the measure* μ if the Oseledets subspaces are continuous, in measure, as functions of the cocycle. More precisely, for any sequence $\{(A_k)_{k\in\mathbb{N}}\} \subset H^r(M)$ converging to A in the r-Hölder topology and any $\varepsilon > 0$, we have

$$\mu\Big(\Big\{x \in M;\ d(E_x^{u,A_k}, E_x^{u,A}) < \varepsilon \quad \text{and} \quad d(E_x^{s,A_k}, E_x^{s,A}) < \varepsilon\Big\}\Big) \xrightarrow{k\to\infty} 1.$$

Thus, our main result is the following

Theorem 2.1. *If $A \in H^r(M)$ is a fiber-bunched cocycle with $\lambda^+(A,\mu) > \lambda^-(A,\mu)$ then it is a continuity point for the Oseleteds decomposition with respect to the measure μ.*

The hypotheses that A is fiber-bunched and μ has local product structure are only used to apply the results about continuity of Lyapunov exponents from [3]. Thus, more generally, if we have a sequence $\{(A_k)_{k\in\mathbb{N}}\} \subset H^r(M)$ *converging uniformly with holonomies* to A as in the main theorem of [3], then

$$\mu\Big(\Big\{x \in M;\ d(E_x^{u,A_k}, E_x^{u,A}) < \varepsilon \quad \text{and} \quad d(E_x^{s,A_k}, E_x^{s,A}) < \varepsilon\Big\}\Big) \xrightarrow{k\to\infty} 1.$$

Consequently, our result also applies if we restrict ourselves to the space of locally constant cocycles endowed with the uniform topology.

3 Proof of the theorem

Let us consider the *projective cocycle* $F_A : M \times \mathbb{P}(\mathbb{R}^2) \to M \times \mathbb{P}(\mathbb{R}^2)$ associated to A and f which is given by

$$F_A(x,v) = (f(x), \mathbb{P}A(x)v)$$

where $\mathbb{P}A$ denotes the action of A on the projective space. We say that an F_A-invariant measure m on $M \times \mathbb{P}(\mathbb{R}^2)$ *projects* to μ if $\pi_* m = \mu$ where $\pi : M \times \mathbb{P}(\mathbb{R}^2) \to M$ is the canonical projection on the first coordinate. Given a non-zero element $v \in \mathbb{R}^2$ we are going to use the same notation to denote its equivalence class in $\mathbb{P}(\mathbb{R}^2)$.

Let $\mathbb{R}^2 = E_x^{s,A} \oplus E_x^{u,A}$ be the Oseledets decomposition associated to A at the point $x \in M$. Consider also

$$m^s = \int_M \delta_{(x, E_x^{s,A})} d\mu(x)$$

and

$$m^u = \int_M \delta_{(x, E_x^{u,A})} d\mu(x)$$

which are F_A-invariant probability measures on $M \times \mathbb{P}(\mathbb{R}^2)$ projecting to μ. Moreover, by the Birkhoff ergodic theorem and (2.1) we have that

$$\lambda^-(A,\mu) = \int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_A(x,v) dm^s(x,v)$$

and

$$\lambda^+(A,\mu) = \int_{M \times \mathbb{P}(\mathbb{R}^2)} \varphi_A(x,v) dm^u(x,v)$$

where $\varphi_A : M \times \mathbb{P}(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$\varphi_A(x,v) = \log \frac{\| A(x)v \|}{\| v \|}.$$

By the (non-uniform) hyperbolicity of (A,μ) we have the following.

Lemma 3.1. *Let m be a probability measure on $M \times \mathbb{P}(\mathbb{R}^2)$ that projects down to μ. Then, m is F_A-invariant if and only if it is a convex combination of m^s and m^u for some f-invariant functions $\alpha, \beta : M \to [0,1]$ such that $\alpha(x) + \beta(x) = 1$ for every $x \in M$.*

Proof. One implication is trivial. For the converse one only has to note that every compact subset of $\mathbb{P}(\mathbb{R}^2)$ disjoint from $\{E^u, E^s\}$ accumulates on E^u in the future and on E^s in the past. □

Proof of Theorem 2.1. Suppose that A is a fiber-bunched cocycle such that $\lambda^+(A,\mu) > \lambda^-(A,\mu)$. As the subset of fiber-bunched elements of $H^r(M)$ is open we may assume without loss of generality that A_k is fiber-bunched for every $k \in \mathbb{N}$. Moreover, since the Lyapunov exponents depend continuously on the cocycle A (see Theorem 1.1 from [3]) and $\lambda^+(A,\mu) > \lambda^-(A,\mu)$ we may also assume that $\lambda^+(A_k,\mu) > \lambda^-(A_k,\mu)$ for every $k \in \mathbb{N}$. We will prove just the assertion about the unstable spaces, that is, that $\mu\left(\left\{x \in M;\ d(E_x^{u,A_k}, E_x^{u,A}) < \delta\right\}\right) \xrightarrow{k \to \infty} 1$. The case of the stable spaces is analogous.

For each $k \in \mathbb{N}$, let us consider the measure

$$m_k = \int_M \delta_{(x, E_x^{u,A_k})} d\mu(x)$$

and let m^u be the measure given by

$$m^u = \int_M \delta_{(x,E_x^{u,A})} d\mu(x).$$

These are F_{A_k} and F_A-invariant probability measures on $M \times \mathbb{P}(\mathbb{R}^2)$ respectively, projecting to μ. Moreover, $m_k \xrightarrow{k\to\infty} m^u$. Indeed, let $(m_{k_j})_{j\in\mathbb{N}}$ be a convergent subsequence of $(m_k)_{k\in\mathbb{N}}$ and suppose that it converges to η. Since for each $j \in \mathbb{N}$ the measure m_{k_j} is $F_{A_{k_j}}$-invariant and projects to μ it follows that η is an F_A-invariant measure projecting to μ. Moreover, since

$$\lambda^+(A_{k_j}, \mu) \xrightarrow{j\to\infty} \lambda^+(A, \mu)$$

once the Lyapunov exponents are continuous as functions of the cocycle (see [3]) and

$$\lambda^+(A_{k_j}, \mu) = \int_{M\times\mathbb{P}(\mathbb{R}^2)} \varphi_{A_{k_j}} dm_{k_j} \xrightarrow{j\to\infty} \int_{M\times\mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta$$

we get that

$$\lambda^+(A, \mu) = \int_{M\times\mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta.$$

Thus, invoking Lemma 3.1 and using the fact that μ is ergodic it follows that $\eta = m^u$. Indeed, otherwise we would have $\eta = \alpha m^s + \beta m^u$ with $\alpha > 0$ and consequently

$$\int_{M\times\mathbb{P}(\mathbb{R}^2)} \varphi_A d\eta = \alpha\lambda^-(A, \mu) + \beta\lambda^+(A, \mu) < \lambda^+(A, \mu).$$

Therefore, $m_k \xrightarrow{k\to\infty} m^u$ as claimed.

Let $g : M \to \mathbb{P}(\mathbb{R}^2)$ be the measurable map given by

$$g(x) = E_x^{u,A}.$$

Note that its graph has full m^u-measure. By Lusin's Theorem, given $\varepsilon > 0$ there exists a compact set $K \subset M$ such that the restriction g_K of g to K is continuous and $\mu(K) > 1 - \varepsilon$. Now, given $\delta > 0$, let $U \subset M \times \mathbb{P}(\mathbb{R}^2)$ be an open neighborhood of the graph of g_K such that

$$U \cap (K \times \mathbb{P}(\mathbb{R}^2)) \subset U_\delta$$

where

$$U_\delta := \{(x, v) \in K \times \mathbb{P}(\mathbb{R}^2);\ d(v, g(x)) < \delta\}.$$

By the choice of the measures m_k,

$$m_k(U_\delta) = \mu(\{x \in K;\ d(E_x^{u,A_k}, E_x^{u,A}) < \delta\}). \tag{3.2}$$

Now, as $m_k \xrightarrow{k\to\infty} m^u$ it follows that $\liminf m_k(U) \geq m^u(U) > 1 - \varepsilon$. On the other hand, as $m_k(K \times \mathbb{P}(\mathbb{R}^2)) = \mu(K) > 1 - \varepsilon$ for every $k \in \mathbb{N}$, it follows that

$$m_k(U_\delta) \geq m_k(U \cap (K \times \mathbb{P}(\mathbb{R}^2))) \geq 1 - 2\varepsilon \tag{3.3}$$

for every k large enough. Thus, combining (3.2) and (3.3), we get that $\mu(\{x \in M;\ d(E_x^{u,A_k}, E_x^{u,A}) < \delta\}) \geq 1 - 2\varepsilon$ for every k large enough completing the proof of Theorem 2.1.

□

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