# A brief survey on tangent $k$-fields on manifolds *

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**Abstract**

We present a summary of the principal results on the index of a tangent $k$-field with finite singularities defined on a manifold. If the manifold is a closed connected smooth oriented one and the index depends only on the oriented homotopy type of the manifold, such index is expressed precisely in terms of the generators of the 2-primary components of the homotopy groups in dimension $n-1$ of the real Stiefel manifold of k-frames in $\mathbb{R}^n$, where $n$ is the dimension of the manifold. In the non-orientable case, the index is defined as a equivalent class belonging to a certain quotient group.

## 1 Introduction

The subject of indices of tangent $k$-fields with finite singularities defined on manifolds has been very much studied, specially in the second half of last century. Our purpose is to write a simple survey for those readers who would like to be acquainted with this research area, presenting a summary of the classical results and pointing out the relevant references. A good introduction for the subject can be found in [21]. For a reference written in Portuguese see [18]. The general idea is to use algebraic invariants of the manifold to detect its geometric property of admitting a tangent $k$-field with finite singularities or without singularities. First one has to deal with the problem of the existence a of tangent $k$-field with finite singularities defined on the manifold. If this is possible, then one can define the index of this $k$-field and it measures the possibility of removing the singularities. It must be null in order to remove the singularities [20]. The index is the obstruction to obtain a $k$-field without singularities on the manifold.

In section 2 we present the definition of the index of a tangent $k$-field defined on a closed connected smooth manifold, for the case when the manifold is oriented and the case when the manifold is non-orientable [3], [20], [18].

In section 3 we give a brief history of the theory of $k$-fields, presenting the main results and their respective references.

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We will omit the proves of the results presented, for the technical details can be found in the cited references.

## 2 Index of a tangent $k$-field with finite singularities

Let $M$ be a closed connected oriented smooth manifold of dimension $n$ and, for each $x \in M$, let us denote by $T_xM$ the tangent space to the manifold $M$ at $x$ and by $(TM, p, M)$ the tangent bundle of $M$, where $TM = \bigcup_{x\in M}(T_xM)$ and $p : TM \to M$ is the projection $p(x, v_x) = x$, $v_x \in T_xM$.

**Definition 2.1 (continuous tangent field)** *A continuous tangent field defined on $M$ is section of the tangent bundle of the manifold $M$. It is a continuous function $s : M \to TM$, such that $p \circ s = id_M$.*

**Definition 2.2 (tangent $k$-field)** *A continuous tangent $k$-field defined on a manifold $M$ of dimension $n$, or simply a $k$-field on $M$, where $1 \leq k \leq n$, is an ordered $k$-tuple $u = (u_1, u_2, \ldots, u_k)$ such that each $u_i, 1 \leq i \leq k$, is a section of the tangent bundle of M. That is, each $u_i, 1 \leq i \leq k$, is a continuous tangent vector field defined on $M$.*

**Definition 2.3 (singularity set)** *Let $u = (u_1, u_2, \ldots, u_k)$ be a $k$-field defined on $M$. The singularity set of $u$, denoted by $S(u)$ is the set*

$$S(u) = \{x \in M \mid u_1(x), \ldots, u_k(x) \text{ are linearly dependent tangent vectors}\}.$$

*When $S(u) = \emptyset$ the $k$-field is said to be a $k$-field without singularities. If $S(u)$ is a finite set, the $k$-field is said to be finitely singular or with finite singularities.*

**Remark 2.1** *The singularity set does not need to be finite. As an example take $M = S^1 \times D^2$, where $S^1 \subset \mathbb{R}^2$ is the unidimensional sphere and $D^2 \subset \mathbb{R}^2$ is the unitary 2-disk, having the origin as its center. Define a tangent vector field without singularities on $S^1$ by $u_1 : S^1 \to T(S^1), u_1(x_1, x_2) = ((x_1, x_2), (-x_2, x_1))$ and a radial tangent vector field on $D^2$, $u_2 : D^2 \to T(D^2)$, $u_2((y_1, y_2)) = ((y_1, y_2), \lambda(y_1, y_2)), \lambda \neq 0$. The unique singularity of this radial field is the origin. Consider the 2-field on $M = S^1 \times D^2$, defined by $u((x_1, x_2), (y_1, y_2)) = (u_1(x_1, x_2), u_2(y_1, y_2))$. The singularity set $S(u) = \{((x_1, x_2), (0, 0)) \in S^1 \times D^2\}$ is homeomorphic to $S^1$.*

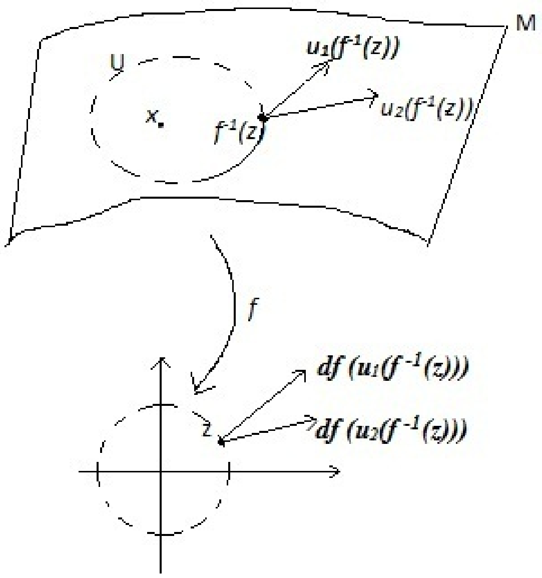
Let us suppose that a closed connected smooth manifold admits a $k$-field with finite singularities. The index of the $k$-field $u$, denoted by $ind(u)$, is the obstruction to obtain a $k$- field defined on $M$ without singularities. That is, the index of a $k$-field defined on $M$ is the obstruction to remove the singularities. In order to define $ind(u)$, we will consider the real Stiefel manifold of k-frames in $\mathbb{R}^n$, denoted by $V_{n,k}$. The index of the $k$-field $u$ will be expressed in terms of the generators of the homotopy group $\pi_{n-1}(V_{n,k})$ [1], [20]. It is important to know when the index of a $k$-field defined on $M$ does not depend on the particular $k$-field $u$, but only on the oriented homotopy type of the manifold $M$.

## 2.1 Index of a tangent $k$-field with finite singularities defined on an oriented manifold

Let $M$ be a closed connected smooth manifold of dimension $n$ and suppose that $M$ admits a $k$-field $u = (u_1, u_2, \ldots, u_k)$ with finite singularities. Let $x \in S(u)$ and consider a local chart $(U, f)$, $f : U \to \mathbb{R}^n$, for the singularity $x$, such that $f(x) = 0$, $f(U) \supset D^2$ and $U \cap S(u) = \{x\}$. Recall that a chart $(U, f)$ induces a local orientation in the neighborhood $U$ of the singularity $x$. Let $V_{n,k}$ denote the real Stiefel manifold of k-frames in $\mathbb{R}^n$.

**Definition 2.4 (local index of the $k$-field $u$ at a singularity)** *The local index of the $k$-field $u$ at the singularity $x$ is the homotopy class of the map $f_x : S^{n-1} \to V_{n,k}$, defined by*

$$f_x(z) = (df(u_1(f^{-1}(z)), df(u_2(f^{-1}(z)), \ldots, df(u_k(f^{-1}(z))))$$




Notice that the $k$-tuple $(df(u_1(f^{-1}(z)) \ldots df(u_k(f^{-1}(z))))$ is a $k$-frame in $\mathbb{R}^n$.
The local index of the $k$-field $u$ at the singularity $x$ is

$$ind_x(u) = [f_x] \in \pi_{n-1}(V_{n,k})$$

In order to simplify the notation above, we have omitted the point $f^{-1}(z)$, where the differential $df$ is evaluated, that is, we are denoting $df_{f^{-1}(z)}(u_i(f^{-1}(z))$ simply by $df(u_i(f^{-1}(z)), 1 \leq i \leq k$.

**Remark 2.2** *The local index of a $k$-field at a singularity $x$ depends on the choice of the chart, once the chart induces a local orientation on the neighborhood $U$. But if we chose two charts $(U, f)$ and $(V, g)$, such that $x \in U \cap V$, both inducing the same local orientation in a neighborhood of $x$, then the local index* *of a $k$-field, $u$, defined on the manifold $M$, at a singularity $x \in M$, is independent of the chart chosen to define it. Now let us consider an orientable manifold $M$ and let us fix an orientation on $M$. Any chart $(U, f)$ belonging to the differentiable structure of the oriented manifold $M$ induces a local orientation which is compatible with the fixed orientation of the manifold. So the local index for oriented manifolds is well defined. The properties of the local index of $k$-fields can be found in [20], [18].*

**Definition 2.5 (index of the $k$-field $u$ defined on an oriented manifold)** *The index of the $k$-field with finite singularities, $u$, defined on a closed connected oriented smooth manifold of dimension $n$ is given by*

$$ind(u) = \sum_{x \in S(u)} ind_x(u) \in \pi_{n-1}(V_{n,k})$$

**Example 2.1** *Let $M^3$ be a closed oriented manifold of dimension 3. The index of any 3-field $u = (u_1, u_2, u_3)$ with finite singularities is null. So, $M$ admits a 3-field without singularities, this is $M$ is parallelizable. In fact, $ind(u) \in \pi_2(V_{3,3}) = \pi_2(O(3))$. From the induced homotopy exact sequence of the fibration $SO(3) \xrightarrow{i} O(3) \to \mathbb{Z}_2$, where $i$ is inclusion, we have that*

$$\pi_2(O(3)) \cong \pi_2(SO(3)) \cong \pi_2(\mathbb{RP}^3) \cong \pi_2(S^3) = 0$$

*So, $\pi_2(O(3)) = 0$ and $ind(u)$ has to be null.*

**Remark 2.3** *Although the index of a $k$-field $u$ with finite singularities on a closed oriented manifold belongs to $\pi_{n-1}(V_{n,k})$, not all elements of this homotopy group might occur as the index of a $k$-field. By Koschorke [11], if $2k + 1 < n$, $n =$dim $M$, then $ind(u)$ belongs to the image of the projection $p : \pi_{n-1}(V_{n,k+1}) \to \pi_{n-1}(V_{n,k})$. Now, considering the projection of 8-fields to 7- fields, $p : \pi_{n-1}(V_{n,8}) \to \pi_{n-1}(V_{n,7})$, there is an element $\alpha \in \pi_{n-1}(V_{n,7})$ that does not belong to $p(\pi_{n-1}(V_{n,8}))$. So, there is not any 7-field $u$ such that $ind(u)$ is equal to $\alpha$ (see [4]). Using Nomuras notation [16], this element $\alpha$ is equal to $i_3[\nu]$.*

## 2.2 Index of a tangent $k$-field with finite singularities defined on non-orientable manifold

Let $M$ be an orientable manifold and consider the two possible orientations on $M$. If we use a chart inducing one local orientation on a neighborhood of the singularity of $x$ and another one inducing the opposite local orientation, then the indices, $[f]$ and $[\widetilde{f}]$, defined using one chart and the other one that induces the opposite local orientation differs. They are related by $[\widetilde{f}] = -\mu_\star[f]$, where $\mu_\star$ is the homotopy class of the involution $\mu : V_{n,k} \to V_{n,k}$ that changes the sign of a row of a $n \times k$-matrix which represents an element of the Stiefel manifold $V_{n,k}$. See [10] and [18] for further informations about the involutions defined on Stiefel Manifolds.

Let us now consider a non-orientable manifold and the quotient group:

$$\pi(n, k) = \pi_{n-1}(V_{n,k})/(1 + \mu)\pi_{n-1}(V_{n,k})$$

The image of $ind(u)$ in the quotient group $\pi(n,k)$ is independent of the the system of local orientations used to define the $ind(u)$. So, the index of a $k$-field $u$ defined on a non-orientable closed connected smooth manifold is defined as

$$I = [ind(u)] = ind(u) + (1+\mu)\pi_{n-1}(V_{n,k})$$

# 3 A brief history of $k$-fields

Let $M^n$ be a closed connected smooth manifold of dimension $n$ and suppose that it admits a continuous tangent $k$-field $u = (u_1, \ldots, u_k)$, for $1 \leq k \leq n$, with finite singularities. Our aim is to study the possibility of defining a continuous tangent $k$-field without singularities, also called a $k$-frame, on the manifold. The index of the $k$-field is the obstruction to remove the singularities and it is given by means of an algebraic invariant of the manifold. So, if the index of the $k$-field is null it means that it is possible to define a continuous tangent $k$-field without singularities on the manifold.

1. For $k = 1$.
   When $k = 1$ we have the classical result known as Poincaré-Hopf Theorem that says that the index of a continuous tangent vector field with finite singularities defined on a closed oriented manifold is independent of the choice of vector field and it is equal to $ind(u) = \chi(M)\iota_n \in \pi_{n-1}(S^{n-1})$, where $\chi(M)$ is the Euler characteristic of the manifold $M$ and $\iota_n$ is the generator of $\pi_{n-1}(S^{n-1})$, which is isomorphic to $\mathbb{Z}$. Thus, such index, which is the sum of the local indices at each singularity, is identified with the Euler characteristic, $\chi(M)$, and the manifold admits a continuous tangent vector field if and only if its Euler characteristic is null (see [8], [15]). Jules Henri Poincaré (1854-1912) proved the theorem for manifolds of dimension two. In 1926, Heinz Hopf (1894-1971) generalized the result for manifolds of higher dimensions. For history issues see [23].

2. For $k = 2$, M oriented.
   The problem of determining the index of a 2-field with finite singularities defined on a closed oriented connected smooth manifold was completely studied by Thomas [21], [22], F. Hirzebruch and H. Hopf [9], Atiyah and Dupont [2], [3], Frank [7] and Mahowald [13]. Such a index belongs to $\pi_{n-1}(V_{n,2})$, which is isomorphic to $\mathbb{Z}_2$, for $n$ odd and isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, for $n$ even. The computations of the index of a 2-field with finite singularities are presented in table 1 of [21] and table 15 of [4].

   For $k = 2, n = 2$, and $n = 3$, if $M$ is an oriented manifold, then $M$ admits a 2-field with finite singularities. For $dim(M) = 2$, $M$ an oriented manifold, the index of the 2-field is independent of the particular 2-field and $indu = \chi(M)$ [21]. For $dim(M) = 3$, $M$ an oriented manifold, example 2.1 shows that it is parallelizable. So in particular, $M$ has a 2-field without singularities, that is $ind(u) = 0$.

For $k = 2$, $n = 4$, $M$ an oriented manifold, F. Hirzebruch and H. Hopf [9] proved that $ind(u)$ is independent of the $k$-field $u$ if and only if the homology group $H_2(M, \mathbb{Q}) = 0$.

For $k = 2, n > 4$, $M^n$ an oriented manifold, M. F. Atiyah and J. L. Dupont [2], [3] proved that $ind(u)$ is independent of the $k$-field $u$. It depends only on the oriented homotopy type of the manifold.

3. For $k = 2$, $M^n$ nonorientable.
   The index of a 2-field on a nonorientable manifold is not necessarily independent of the choice of the particular 2-field.

   According to Thomas [21], if n is odd, ${w_1}^2(M) = 0$, the index of a 2-field is independent of the choice of the 2-field. But, if n is odd, ${w_1}^2(M) \neq 0$, for any class $u \in H^{n-2}(M, \mathbb{Z}_2)$, one can find a 2-field $u = (u_1, u_2)$ with finite singularity, such that

   $$ind(u) = ({w_1}^2(M).u)[M] \in \mathbb{Z}_2.$$

   For further references for the case $k = 2$ and $M$ a non-orientable manifold see Pollina [17], who describes the obstructions to a tangent 2-field defined on an even dimensional nonorientable manifold and also [14], who treated a more general problem, that includes the existence of 2-fields on nonorientable manifolds as a particular case.

4. For $k = 3$.
   For closed oriented manifolds there are results of Atiyah and Dupont [3], [6] and for closed nonorientable manifolds there are results of Koschorke [11], [12] and Randall [20]. A good summary of the necessary and sufficient conditions for the existence of a 3-field defined on oriented as well as on nonorientable manifolds can be found in [20].

   For $k = 3$, $n = 4$, $M$ an oriented manifold, the result already cited of Hirzebruch and Hopf [9] is still true. The index of a 3-field with finite singularities is independent of the $k$-field $u$ if and only if the homology group $H_2(M, \mathbb{Q}) = 0$.

   $k = 3, n > 5$, $M^n$ an oriented manifold, M. F. Atiyah and J. L. Dupont [2], [3] proved that $ind(u)$ is independent of the $k$-field $u$. It depends only on the oriented homotopy type of the manifold.

5. For $k = 4$.
   Results for 4-fields on closed oriented manifolds can be found in [11], [19], [1] and [4]. Arraut and

Randall proved in [3] that when dim(M) $= n \equiv 3 \text{mod } 4$ the index of a 4-field with finite singularities, $u$, defined on a closed oriented manifold may depend on the 4-field $u$. Nevertheless, when the Manifold has dimension odd and it is a spin one, that is $w_2(M) = 0$, where $w_2(M)$ is the second Sitifel-Whitney class of $M$, then the index of a 4-field is independent of the 4-field $u$. A complete table for the index of a 4-field defined on a closed oriented manifold can be found in [4].

6. For $k = 5$ and $k = 6$.
   Using the isomorphim $p_5 : \pi_{m-1}(V_{m,6}) \longrightarrow \pi_{m-1}(V_{m,5})$, it is clear that the indices of 6-fields are similar to the indices of 5-fields. See [5].

We remark that the projections of $k$-fields with finite singularities to $r$-fields with finite singularities, for $r < k$, allows us to use the already known indices of lower dimensions, $r$, to determine the indices of $k$-fields for higher dimension of $k$. Many results for the index of a $k$-field with finite singularities, for $k > 6$ follow easily using this technique. The computations depend on the equivalent class of $n$ mod $k$. In some cases difficult computations may arise. It is also important to ask when the index of a $k$-field is independent of the given $k$-field. Thus, for $k > 6$ the problem of determining the indices of a $k$-field with finite singularities remains open in many cases, depending on the dimension of $k$ and the equivalent class of $n$ mod $k$.

Different methods have being used to compute the index of a $k$-field, mainly index theory by Atiyah and Dupont, K-theory by Frank and Randall, Postnikov tower resolution by Thomas and others and the analysis of the obstruction using normal bordism theory by Koschorke.

The defitions of index of a $k$- field with finite singularities can also be applied to compact manifolds with boundary. In that case, the $k$-field must obey some property when restrict to the boundary, $\partial M$, of the manifold and it is supposed that $S(u) \subset M - \partial M$. As a reference for $k$-fields defined on manifolds with boundary see [3]. If the manifold is not compact and it admits a $k$-field with finite singularities all of them being isolated singularities, the definition the index of the $k$-field still applies (see [18]).

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