HIDDEN REGULARITY FOR A HYPERBOLIC EQUATION
WITH A RESISTANCE TERM

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Abstract

In this paper we obtain the hidden regularity for the weak solution of the nonlinear equation whose linear model is originated from a problem of elasticity with a resistance of the material.

Resumo

Neste artigo obtemos a Regularidade Escondida para a solução fraca de uma equação não-linear, cujo modelo tem origem em um problema de elasticidade com uma resistência do material.

1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\Gamma$. We represent by $Q$ the cylinder $\Omega \times (0, T)$ with lateral boundary $\Sigma = \Gamma \times (0, T)$, $T$ an arbitrary positive real number. In this article we consider the system

\[
\begin{align*}
&|u'' - \Delta u + F(u) = -\nabla p \quad \text{in} \quad Q, \\
&\text{div } u = 0 \quad \text{in} \quad Q, \\
&u = 0 \quad \text{on} \quad \Sigma, \\
&u(0) = u_0, \quad u'(0) = u_1 \quad \text{in} \quad \Omega,
\end{align*}
\]

(1.1)

that appear in the theory of elasticity of incompressible material. We observe that $u : Q \rightarrow \mathbb{R}^n$, $u = (u_1, u_2, ..., u_n)$, where $u_i : Q \rightarrow \mathbb{R}$ for $i = 1, ..., n$, $\text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$, $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

\[
F(u) = \begin{bmatrix}
    f_1 & 0 & \cdots & 0 \\
    0 & f_2 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & f_n
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_n
\end{bmatrix},
\]

where we assume that each $f_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, ..., n$ is a continuous function that satisfies

\[
f_i' \in L^\infty(\mathbb{R}), \quad \lim_{s \to \infty} \frac{f_i(s)}{s} = \alpha_i, \quad i = 1, ..., n.
\]

(1.2)

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This model, without the nonlinear term, was proposed in the first time in a work of J. L. Lions [6]. The aim of this paper it shows that, under the condition (1.2) and a geometrical condition on Ω, the weak solution of the problem (1.1) has the following regularity on the lateral boundary Σ

\[ \frac{\partial u}{\partial \nu} \in [L^2(\Sigma)]^n, \]  

(1.3)

where \( \nu \) is the exterior normal vector to \( \Gamma \). The property (1.3) was proved by J. L. Lions [6], motivated by problems of optimal control for the equation \( u'' - \Delta u = 0 \) in \( Q \). In [7], J. L. Lions proves that the weak solution \( u \) of the mixed problem for the equation

\[ u'' - \Delta u + |u|^\rho u = 0 \text{ in } Q, \]

also satisfies (1.3) and called this property of "Hidden Regularity", because that one does not follow from the equation or from the spaces where the solution lies. This result was generalized by M. Milla Miranda and L. A. Medeiros [8] for the equation

\[ u'' - \Delta u + F(u) = 0 \text{ in } Q, \]

where \( F : \mathbb{R} \rightarrow \mathbb{R} \) satisfies the hypothesis of W. A. Strauss [10], i.e., \( F \) is continuous and \( sF(s) \geq 0, s \in \mathbb{R} \). In [11], M. C. de Campos Vieira obtain the Hidden Regularity for the following Coupled System of Nonlinear Hyperbolic Equations

\[ u'' - \Delta u + \alpha |u|^p + 2 |u|^\rho u = g_1 \text{ in } Q \]

\[ v'' - \Delta v + \alpha |u|^p + 2 |v|^\rho v = g_2 \text{ in } Q. \]

The Hidden Regularity for the linear system associated to (1.1) was proved in [5] by Lions. For a physical interpretation of this model see Rocha dos Santos [9]. See about similar questions in Kapitonov [3], [4].
2 Existence and Uniqueness of Solution

In this section we study the existence and uniqueness of solution for the problem (1.1) that will be written in the following form:

\[
\begin{align*}
    u_i^\prime - \Delta u_i + f_i(u_i) &= -\frac{\partial p}{\partial x_i} \quad \text{in } Q, \\
    \sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j} &= 0 \quad \text{in } Q, \\
    u_i &= 0 \quad \text{on } \Sigma, \\
    u_i(0) &= u_{0i}, \quad u_i'(0) = u_{1i} \quad \text{in } \Omega,
\end{align*}
\]

for \( i = 1, \ldots, n \), where the prime means the derivative with respect to \( t \) and all the derivatives are in the sense of the theory of distributions.

Before beginning the study of (2.4), the notation will be established. We consider the Hilbert spaces

\[
V = \left\{ u \in \left( H^1_0(\Omega) \right)^n \mid \text{div } u = 0 \right\}
\]

and

\[
H = \left\{ u \in \left( L^2(\Omega) \right)^n \mid \text{div } u = 0 \text{ and } u.\nu = 0 \text{ on } \Gamma \right\},
\]

equipped with the inner product and norm given respectively by

\[
(u, v) = \sum_{i=1}^{n} (u_i, v_i)_{H^1_0(\Omega)}, \quad \|u\|^2 = \sum_{i=1}^{n} \|u_i\|^2_{H^1_0(\Omega)}
\]

and

\[
(u, v) = \sum_{i=1}^{n} (u_i, v_i)_{L^2(\Omega)}, \quad |u|^2 = \sum_{i=1}^{n} |u_i|^2_{L^2(\Omega)}.
\]

**Remark 2.1** Note that it is necessary to show that for each \( w \in H^1_0(\Omega) \), \( f_i(w) \in L^2(\Omega) \). In fact, from the hypothesis (1.2) follows that given \( \varepsilon = 1 \) there exists \( N_i = N_i(\varepsilon) \) such that

\[
|f_i(w) - \alpha_i w| < |w|, \quad \forall \ |w| > N_i,
\]

this is,

\[
|f_i(w)|^2 < 2 \left( |\alpha_i w|^2 + |w|^2 \right), \quad |w| > N_i.
\]

From (2.5) we obtain

\[
\int_{\Omega} |f_i(w)|^2 \, d\Omega < \int_{|w| \leq N_i} |f_i(w)|^2 \, d\Omega + \int_{|w| > N_i} |f_i(w)|^2 \, d\Omega < C_i + 2 \left( |\alpha_i|^2 + 1 \right) \int_{\Omega} |w|^2 \, d\Omega,
\]

because \( f_i \) is continuous and \( \Omega \) is bounded.
Theorem 2.1 Given $u_0 \in V$, $u_1 \in H$ and $f_i$ satisfying (1.2), there exists a unique function $u$ such that

\begin{align}
  u &\in L^\infty(0,T;V), \\
  u' &\in L^\infty(0,T;H), \\
  u'' - \Delta u + F(u) &= -\nabla p \quad \text{in} \quad \mathcal{D}'\left(0,T;\mathcal{D}'(\Omega)\right)^3, \tag{2.6} \\
  u(0) &= u_0, \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \tag{2.7}
\end{align}

Before beginning the proof of the Theorem 2.1, let us consider the following lemma:

Lemma 2.1 Considering $u_0$, $u_1$ and $f_i$ in the conditions of the Theorem 3.1. Then there exists a unique function $u$ satisfying

\begin{align}
  u &\in L^\infty(0,T;V), \\
  u' &\in L^\infty(0,T;H), \\
  \frac{d}{dt}(u'(t), v) + ((u(t), v)) + (f_i(u(t)), v) &= 0, \forall v \in V \quad \text{in} \quad \mathcal{D}'(0,T), \tag{2.10} \\
  u(0) &= u_0, \quad u'(0) = u_1 \quad \text{in} \quad \Omega. \tag{2.11}
\end{align}

Proof. We employ the Faedo-Galerkin’s method. Let $\{w_\nu\}_{\nu \in \mathbb{N}}$ a Hilbertian basis of $V$. We consider $V_m = [w_1, ..., w_m]$ the subspace of $V$ generated by the $m$ first vectors of $\{w_\nu\}_{\nu \in \mathbb{N}}$. The approximate problem of (2.12) − (2.13) consists in to find $u_m(t) \in V_m$ satisfying for each $i = 1, ..., n$

\begin{align}
  \frac{d}{dt}(u'_m(t), v) + ((u_m(t), v)) + (f_i(u_m(t)), v) &= 0, \forall v \in V_m, \\
  u_m(0) &= u_{0m} \rightarrow u_0 \quad \text{strong in} \quad H_0^1(\Omega), \\
  u'_m(0) &= u_{1m} \rightarrow u_1 \quad \text{strong in} \quad L^2(\Omega). \tag{2.12}
\end{align}

By Carathéodory’s theorem, (2.14) has solution on an interval $[0, t_m]$, $t_m < T$. This solutions extend to the whole interval $[0, T]$ as a consequence of the estimates that shall be obtained in the next step.

Estimates. Taking $v = u'_m(t)$ in (2.14), we have after some calculations that

\begin{align}
  (u_m) \quad \text{is bounded in} \quad L^\infty(0,T;V) \tag{2.15}
\end{align}

and

\begin{align}
  (u'_m) \quad \text{is bounded in} \quad L^\infty(0,T;H), \quad i = 1, ..., n. \tag{2.16}
\end{align}

To obtain the convergence of the nonlinear term we observe that from Theorem of Lions-Aubin there exists a subsequence $(u_{im})$ such that:

\begin{align}
  u_{im} \rightarrow u_i \quad \text{strong in} \quad L^2(0,T;L^2(\Omega)). \tag{2.17}
\end{align}

From (2.17) and the continuity of the $f$ follows that

\begin{align}
  f_i(u_{im}(x,t)) \rightarrow f_i(u_i(x,t)) \quad \text{a.e. in} \quad Q. \tag{2.18}
\end{align}
We also have, from Remark 1, that \( f_i(u_{im}) \in L^2(Q) \), for \( i = 1, 2, ..., n \). Therefore, by Lemma of Lions we get
\[
f_i(u_{im}) \rightharpoonup f_i(u_i) \quad \text{in} \quad L^2(0, T; L^2(\Omega)) \quad \text{weak as} \quad m \to +\infty, \quad \text{for} \quad 1 \leq i \leq n.
\] (2.19)

From (2.15), (2.16) and the convergence in (2.19), it is permissible to pass to the limit in (2.14) and obtain, by standard procedure, a solution \( u \) satisfying (2.10) – (2.13).  

**Proof of Theorem 3.1.** To prove the Theorem 2.1, it is sufficient to prove that \( u \) satisfies (2.10) – (2.13) if, and only if, \( u \) satisfies (2.6) – (2.9). In fact, we consider the following functional:
\[
L(v) = \langle u'' - \Delta u + F(u), v \rangle_{(H^{-1}(\Omega))^n \times (H^1_0(\Omega))^n},
\]
where \( u \) is the solution given by the Lemma 2.1. Observe that \( L \in (H^{-1}(\Omega))^n \). From (2.12) we have
\[
L(v) = 0 \quad \text{for all} \quad v \in V, \quad \text{in} \quad D'(0, T). \quad \text{By the De Rham's lemma [2], there exists} \quad p \in L^2(\Omega) \setminus \mathbb{R}
\]
such that
\[
L(v) = \int_\Omega p \text{div} \, v \, dx, \quad \forall v \in V.
\]
In particular, for all \( \varphi \in V = \{ \varphi \in (D(\Omega))^n ; \text{div} \, \varphi = 0 \} \). So, we obtain
\[
\langle u'' - \Delta u + F(u), \varphi \rangle = \langle -\nabla p, \varphi \rangle,
\]
is this,
\[
u'' - \Delta u + F(u) = -\nabla p \quad \text{in} \quad D' \left( 0, T; (D' (\Omega))^3 \right).
\]
Reciprocally, if \( u \) satisfies (2.6) – (2.9), then multiplying (2.8) by \( \varphi \in V \) and integrating in \( \Omega \), we conclude that
\[
\langle u'' (t), \varphi \rangle + \langle (u (t), \varphi) \rangle + \langle F (u), \varphi \rangle = -\langle \nabla p, \varphi \rangle, \quad \text{in} \quad D' \left( 0, T \right), \forall \varphi \in V,
\]
thus the theorem is proved.  

**2.1 Energy Conservation**

From the weak formulation of the (2.4), taking \( v = u' \) we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ ||u'_i(t)||^2 + ||u_i(t)||^2 \right] + \langle f_i(u_i(t)), u'_i(t) \rangle = 0. \quad (2.20)
\]

If we represent by \( G_i(s) \) the function
\[
G_i(s) = \int_0^s f_i(r) \, dr
\]
then
\[
\langle f_i(u_i(t)), u'_i(t) \rangle = \frac{d}{dt} \int_\Omega G_i(u_i(t)) \, dx. \quad (2.21)
\]

Substituting (2.21) in (2.20) we obtain
\[
\frac{d}{dt} E(t) = 0,
\]
where \( E(t) = \frac{1}{2} \left[ ||u'_i(t)||^2 + ||u_i(t)||^2 \right] + \int_\Omega G_i(u_i(t)) \, dx \). Therefore \( E(t) = E(0) = E_0 \).
3 Main Result

In what follows we shall prove (1.3) under the assumption
\[ m(x) \cdot \nu \geq \gamma > 0, \quad m(x) = x, \text{ for } x \in \Gamma. \]  

(3.22)

**Theorem 3.1** We assume that (1.2) and (3.22) holds true. Then, if \( u \) denotes the solution of (1.1) with initial data \( \{u_0, u_1\} \in V \times H \), one has
\[ \frac{\partial u}{\partial \nu} \in \left( L^2(\Sigma) \right)^n. \]  

(3.23)

**Proof:** In the follows, let us to prove (3.23) for \( u_0, u_1 \) smooth, this is, \( u_0, u_1 \in U \) where
\[ U = \{ \phi \in \left( D(\Omega) \right)^n \text{ such that } \text{div } \phi = 0 \}. \]

If we denote by \( u_\mu \) the solution associated to \( u_0, u_1 \in U \) such that,
\[ u_0 \rightarrow u_0 \text{ in } V \]
\[ u_1 \rightarrow u_1 \text{ in } H, \]

than we have
\[ u_\mu \in L^\infty \left( 0, T; H^2(\Omega) \cap V \right) \]
\[ u_\mu' \in L^\infty \left( 0, T; V \right) \]
\[ u_\mu'' \in L^\infty \left( 0, T; H \right), \]  

(3.24)

and so the integrations by parts to follow are valid.

Multiply both sides of (2.4) by \( m(x) \cdot \nabla u_\mu = \sum_{k=1}^n m_k \frac{\partial u_\mu}{\partial x_k} \), and integrate on \( Q \), we obtain:
\[
\int_Q \left( u_\mu'' m_k \frac{\partial u_\mu}{\partial x_k} - \Delta u_\mu m_k \frac{\partial u_\mu}{\partial x_k} + f_i (u_\mu) m_k \frac{\partial u_\mu}{\partial x_k} \right) dx dt = -\int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial u_\mu}{\partial x_k} dx dt \]  

(3.25)

**Remark 3.1** If we integrate by parts the last term of the (3.25) we obtain
\[ -\int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial u_\mu}{\partial x_k} dx dt = 0. \]

Now we will analyze each term of the first member of the (3.25).

- \[ \int_Q u_\mu'' m_k \frac{\partial u_\mu}{\partial x_k} dx dt = \left( u_\mu', m_k \frac{\partial u_\mu}{\partial x_k} \right) \bigg|_0^T + \frac{n}{2} \int_Q (u_\mu')^2 dx dt. \]

- \[ -\int_Q \Delta u_\mu m_k \frac{\partial u_k}{\partial x_k} dx dt = -\frac{1}{2} \int_{\Sigma} m_k \cdot \nu_k \left| \frac{\partial u_\mu}{\partial \nu} \right|^2 d\Sigma + \frac{1}{2} \int_Q \left| \frac{\partial u_\mu}{\partial x_k} \right|^2 dx dt. \]

**Remark 3.2** For the analysis of the nonlinear term we use the Strauss approximation cf. [10] and [8] to prove that \( G_i \) is \( C^1 \) with bounded derivative and \( G_i(0) = 0 \), this is, \( G_i(u_i) \in L^2(0, T; H^1_0(\Omega)) \), for \( i = 1, 2, \ldots, n \). Therefore we have,
• \( \int_Q f_i (u_{\mu_i}) m_k \frac{\partial u_{\mu_i}}{\partial x_k} \, dx \, dt = -n \int_Q G_i (u_{\mu_i}) \, dx \, dt. \)

Substituting the estimates above in (3.25) and from (3.22) we obtain, after some calculations that

\[
\frac{\gamma}{2} \int_{\Sigma} \left| \frac{\partial u_{\mu_i}}{\partial \nu} \right|^2 \, d\Sigma \leq n T E_0 + 2n \left| \int_Q G_i (u_{\mu_i}(x,t)) \, dx \, dt \right| + n \left( u'_{\mu_i}(t), m_k \frac{\partial u_{\mu_i}}{\partial x_k} \right) \bigg|_0^T.
\]

From (3.26) follows that

\[
\int_{\Sigma} \left| \frac{\partial u_{\mu_i}}{\partial \nu} \right|^2 \, d\Sigma \leq C (n, T, \Omega) E_0,
\]

because \( \left| \int_Q G_i (u_{\mu_i}(x,t)) \, dx \, dt \right| \leq C (\Omega). \)

Therefore, from (3.27) we have

\[
\frac{\partial u_{\mu_i}}{\partial \nu} \rightarrow \chi \text{ in } L^2 (0,T; L^2 (\Gamma)).
\]

Now observe that by (3.24) and Theorem of Lions-Aubin there exists a subsequence \( (u_{\mu_i}) \) such that

\[
u_{\mu_i} \rightarrow u_i \text{ strong in } L^2 (0,T; H^1 (\Omega)).
\]

By the continuity of the trace we have

\[
\int_0^T \| \gamma_1 (u_{\mu_i}(t)) - \gamma_1 (u_i(t)) \|^2_{H^{-1/2}(\Gamma)} \, dt \leq C \int_0^T \| u_{\mu_i}(t) - u_i(t) \|^2_{H^1(\Omega)} \, dt,
\]

for each \( i = 1, \ldots, n. \)

From (3.29) and (3.30) we conclude that

\[
\frac{\partial u_{\mu_i}}{\partial \nu} \rightarrow \frac{\partial u_i}{\partial \nu} \text{ in } L^2 \left( 0,T; H^{-1/2}(\Gamma) \right) \text{ as } \mu \rightarrow +\infty, \ i = 1, \ldots, n.
\]

From (3.28) and (3.31) follows that

\[
\chi = \frac{\partial u_i}{\partial \nu} \in L^2 (\Sigma), \text{ for each } i = 1, \ldots, n,
\]

this is,

\[
\frac{\partial u}{\partial \nu} \in \left( L^2 (\Sigma) \right)^n.
\]

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References


