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ON CERTAIN MATRICES RELATED TO THE CAUCHY AND TOEPLITZ MATRICES

SOBRE CERTAS MATRIZES RELACIONADAS ÀS MATRIZES DE CAUCHY E TOEPLITZ

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Resumo

Neste artigo, apresentamos uma nova classe de matrizes que não possuem submatrizes singulares sobre um corpo de característica arbitrária. Para fazer isso, primeiro calculamos o determinante de certos tipos de matrizes quadradas, que estão relacionadas às conhecidas matrizes de Cauchy, Toeplitz e Hilbert em caso especial. Também é mostrado que todos os menores dessas matrizes têm posto máximo. Finalmente, aplicando-se nossos resultados, determinamos o posto de uma nova classe de matrizes introduzidas no nosso estudo.

Palavras-chave: Determinante e posto de matrizes, matrizes não singulares, matrizes de Cauchy, Hilbert e Toeplitz.

Abstract

In this paper, we introduce a new class of matrices with no singular submatrices over a field of arbitrary characteristics. To do this end, we first compute the determinant of a certain type of square matrices, which are related to the well-known Cauchy, Toeplitz, and Hilbert matrices in a special case. It is also shown that all of the minors of those matrices have full rank. Then, applying our results, we determine the rank of a newly introduced class of matrices under study.
Keywords: Determinant and rank of matrices, non-singular matrices, Cauchy, Toeplitz and Hilbert matrices.

MSC2010: 15B05, 15A15

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1 Introduction and main result

The Theory of Matrices has an important role in the progresses of fundamentals and researches in different areas of Mathematics as well as Physics and Computer Science. Some classes of matrices, including the Chauchy, Toeplitz, and Hilbert matrices have most applications among the whole theory [2, 3, 4, 8].

In this paper, we will consider a certain class of matrices which are related to the aforementioned classes in a special case and have an interesting property that all of their sub-matrices are non-singular. The matrices with this property have applications in Computer Communications and Coding Theory [6].

Throughout this paper, we fix a field of any characteristic denoted by $\mathbb{F}$. We assume that $\{a_0, \ldots, a_n\}$ is a set of pairwise distinct elements in $\mathbb{F}^*$ for a natural number $n \geq 3$. For any given integer r such that $2 \leq r < n$, we consider the $(n-r) \times (r+1)$ matrix $C := [C_{(i-r,j)}]$ such that for $0 \leq j \leq r$ and $r+1 \leq i \leq n$ the entries $C_{(i-r,j)}$ are given by

$$
\begin{aligned}
C_{(i-r,0)} &= (-1)^r \prod_{\ell \in I} (a_i - a_\ell) \cdot \prod_{e<\ell \in I_0} (a_\ell - a_e) \\
C_{(i-r,j)} &= (-1)^{r+j} a_i \cdot \prod_{\ell \in I_j} a_\ell \cdot \prod_{\ell \in I_j} (a_i - a_\ell) \cdot \prod_{e<\ell \in I_j} (a_\ell - a_e) \\
D_r &= (-1)^r \prod_{\ell \in I} a_\ell \cdot \prod_{e<\ell \in I} (a_\ell - a_e),
\end{aligned}
\tag{1}
$$

where $I = \{0, 1, \ldots, r\}$ and $I_j = I \backslash \{j\}$ for $j \in I$. We define $\mathbf{C}_r^n = [C|D]$ as an $(n-r) \times (n+1)$ blocked matrix, where D is a $(n-r) \times (n-r)$ diagonal matrix with entries D_r. We note that a particular choice of the set of a_i's in characteristic zero shows that the square sub-matrices of C are related to the Cauchy, Toeplitz and Hilbert matrices. For more details, see Remark 1 in Section 2.

The main goal of this paper concerns with proving the following theorem.

Theorem 1.1. *Notation being as above, the followings hold:*

(i) For $m \leq \min\{r+1, n-r\}$, all of the $m \times m$ sub-matrix of $C = [C_{(i-r,j)}]$ are non-singular; hence C has maximal rank equal to $\min\{n-r, r+1\}$.

(ii) The matrix $\mathbf{C}_r^n$ has the full rank $n-r$.

We note that in the positive characteristic case, the above theorem shows that all of the square sub-matrices of C and hence $\mathbf{C}_r^n$ are non-singular. Thus these matrices can be used in Computer Communications and Coding Theory, as in [6]. The readers may found this as an interesting research topic.

The present paper has been organized as follows. In Section 2, we recall the definition and determinant of the some known classes of matrices over fields of any characteristics. In Section 3, we calculate the determinant of a certain square matrix that are related to the Cauchy, Toeplitz and Hilbert matrices. Finally, in Section 4, we use the result of Section 3 to prove Theorem 1.1.

2 Some well-known classes of matrices

In this section, we recall three well-known classes of matrices which are widely used in the different branches of Mathematics.

2.1 Cauchy Matrix

In 1841, Augustin-Louis Cauchy introduced a certain type of matrices with particular properties, see [1, 7]. Here, we recall the definition and determinant of those matrices.

Given a positive integer m and two disjoint sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ of mutually different nonzero elements in $\mathbb{F}$, an $m \times m$ square *Cauchy's matrix* is

$$X_m = [x_{ij}] \text{ where } x_{ij} = \frac{1}{x_i - y_j},\ 1 \leq i, j \leq m.$$

It can be easily seen that any square sub-matrix of a Cauchy's matrix is itself a Cauchy's matrix. The determinant $|X_m|$ of a Cauchy's matrix is known as *Cauchy's determinant* in the literature, which is always nonzero because $x_i \neq y_j$, and can be calculated by,

$$|X_m| = \frac{\prod\limits_{i<j\in\{1,\ldots,m\}} (x_j - x_i)\cdot(y_i - y_j)}{\prod\limits_{i,j\in\{1,\ldots,m\}} (x_i - y_j)}. \tag{2}$$

On can find the above formula in the equation (4) of [9], or the equation (10) in page 154 of [1].

2.2 Hilbert Matrix

In [5], Hilbert introduced a certain square matrix which is a particular case of the Cauchy' square matrix. The *Hilbert's matrix* is an $m \times m$ matrix as follows

$$H_m = [h_{ij}], \text{ with } h_{ij} = 1/(i + j - 1), \ 1 \leq i, j \leq m.$$

Using the formula (2), it is easy to calculate the determinant of a Hilbert's matrix as

$$|H_m| = \frac{c_m^4}{c_{2m}}, \text{ with } c_m = \prod_{i=1}^{m-1} i!.$$

The determinant of H_m is the reciprocal of a well known sequence $\{c_m\}_{m=1}^{\infty}$ of integers (see the sequence A005249 in OEIS [10]) which are defined by the following identity,

$$\frac{1}{|H_m|} = \frac{c_{2m}}{c_m^4} = m! \cdot \prod_{i=1}^{2m-1} \binom{i}{\left[\frac{i}{2}\right]}.$$

2.3 Toeplitz Matrix

The other class of matrices, which we are going to recall here, are called the Toeplitz's matrix introduced by Otto Toeplitz, see [11, 12].

An $m \times m$ *Toeplitz matrix* with entries in $\mathbb{F}$ is a square matrix as

$$T_m := \begin{pmatrix} t_0 & t_1 & t_2 & \cdots & \cdots & t_{m-1} \\ t_{-1} & t_0 & t_1 & \ddots & \ddots & \vdots \\ t_{-2} & t_{-1} & \ddots & \ddots & \ddots & \cdots \\ \vdots & \ddots & \ddots & \ddots & t_1 & t_2 \\ \vdots & \ddots & \ddots & t_{-1} & t_0 & t_1 \\ t_{1-m} & \cdots & \cdots & t_{-2} & t_{-1} & t_0 \end{pmatrix}.$$

The Toeplitz matrices are one of the most well studied and understood classes of matrices which appeared in most areas of Mathematics, say algebra [8], algebraic geometry [3], and graph theory [4].

3 Determinant of a certain square matrix

In this section, we compute the determinant of certain square matrices, which are related to the determinant of Cauchy's matrix. For a given integer $m \geq 1$ and two disjoint subsets $\{x_0, x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ of distinct elements in $\mathbb{F}$, we define the following $(m+1) \times (m+1)$ matrix:

$$Y_m = \begin{pmatrix} 1 & \frac{x_0}{x_0 - y_1} & \frac{x_0}{x_0 - y_2} & \cdots & \frac{x_0}{x_0 - y_m} \\ 1 & \frac{x_1}{x_1 - y_1} & \frac{x_1}{x_1 - y_2} & \cdots & \frac{x_1}{x_1 - y_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{x_m}{x_m - y_1} & \frac{x_m}{x_m - y_2} & \cdots & \frac{x_m}{x_m - y_m} \end{pmatrix}. \tag{3}$$

The following proposition gives a formula to compute the determinant of Y_m.

Proposition 3.1. *For any integer $m \geq 1$, the determinant of Y_m can be computed by*

$$|Y_m| = \frac{(-1)^m \prod_{j=1}^{m} y_j \cdot \prod_{1 \leq i < j \leq m} (x_i - x_j) \cdot \prod_{0 \leq s' < s \leq m} (x_s - x_{s'})}{\prod_{i=0}^{m} \prod_{j=1}^{m} (x_i - y_j)}.$$

Proof. First of all, for any three distinct elements u, v, and w in $\mathbb{F}$, we have:

$$u(v - w) - v(u - w) = -w(u - v). \tag{4}$$

Subtracting the first row of the matrix Y_m from the other ones leads to

$$|Y_m| = \begin{vmatrix} 1 & \frac{x_0}{x_0 - y_1} & \cdots & \frac{x_0}{x_0 - y_m} \\ 0 & \frac{x_1(x_0 - y_1) - x_0(x_1 - y_1)}{(x_0 - y_1)(x_1 - y_1)} & \cdots & \frac{x_1(x_0 - y_m) - x_0(x_1 - y_m)}{(x_0 - y_m)(x_1 - y_m)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{x_m(x_0 - y_1) - x_0(x_m - y_1)}{(x_0 - y_1)(x_m - y_1)} & \cdots & \frac{x_m(x_0 - y_m) - x_0(x_m - y_m)}{(x_0 - y_m)(x_m - y_m)} \end{vmatrix}$$

Then, using the equality (4), we have

$$|Y_m| = \begin{vmatrix} 1 & \dfrac{x_0}{x_0-y_1} & \cdots & \dfrac{x_0}{x_0-y_m} \\ 0 & \dfrac{-y_1(x_1-x_0)}{(x_0-y_1)(x_1-y_1)} & \cdots & \dfrac{-y_m(x_1-x_0)}{(x_0-y_m)(x_1-y_m)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dfrac{-y_1(x_m-x_0)}{(x_0-y_1)(x_m-y_1)} & \cdots & \dfrac{-y_m(x_m-x_0)}{(x_0-y_m)(x_m-y_m)} \end{vmatrix}.$$

By extracting $-y_j/(x_0-y_j)$ from each of the $(j+1)$-th column and $x_i - x_0$ from each of the $(i+1)$-th rows for $1 \le i, j \le m$, we obtain

$$|Y_m| = (-1)^m \prod_{j=1}^{m} \frac{y_j}{(x_0-y_j)} \cdot \prod_{i=1}^{m}(x_i - x_0) \cdot \begin{vmatrix} \dfrac{1}{(x_1-y_1)} & \dfrac{1}{(x_1-y_2)} & \cdots & \dfrac{1}{(x_1-y_m)} \\ \dfrac{1}{(x_2-y_1)} & \dfrac{1}{(x_2-y_2)} & \cdots & \dfrac{1}{(x_2-y_m)} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{(x_m-y_1)} & \dfrac{1}{(x_m-y_2)} & \cdots & \dfrac{1}{(x_m-y_m)} \end{vmatrix},$$

where the last determinant is the Cauchy determinant of a square $m \times m$ Cauchy's matrix X_m, hence using formula (2), we have

$$|Y_m| = (-1)^m \prod_{j=1}^{m} \frac{y_j}{x_0-y_j} \cdot \prod_{i=1}^{m}(x_i - x_0) \cdot \frac{\displaystyle\prod_{1\le s'<s\le m}(x_s - x_{s'}) \cdot \prod_{1\le i<j\le m}(y_i - y_j)}{\displaystyle\prod_{i=1}^{m}\prod_{j=1}^{m}(x_i - y_j)},$$

$$= \frac{\displaystyle(-1)^m \prod_{j=1}^{m} y_j \cdot \prod_{1\le i<j\le m}(y_i - y_j) \cdot \prod_{0\le s'<s\le m}(x_s - x_{s'})}{\displaystyle\prod_{i=0}^{m}\prod_{j=1}^{m}(x_i - y_j)}.$$

□

Remark 1. The determinant of Y_m, in a particular case, is related to the determinant of a certain $m \times m$ Toeplitz and hence the Hilbert matrix. Indeed, considering $x_0 = 0, x_i = 1/i$ and $y_j = 1/(m+j)$ for $1 \le i, j \le m$, and doing some algebraic simplifications, one can get that:

$$Y_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \dfrac{m+1}{m} & \dfrac{m+2}{m+1} & \cdots & \dfrac{2m}{2m-1} \\ 1 & \dfrac{m+1}{m-1} & \dfrac{m+2}{m} & \cdots & \dfrac{2m}{2m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dfrac{m+1}{2} & \dfrac{m+2}{3} & \cdots & \dfrac{2m}{m+1} \\ 1 & m+1 & \dfrac{m+2}{2} & \cdots & \dfrac{2m}{m} \end{bmatrix}.$$

Then, factoring $m+i$ from $(i+1)$-th column for $1 \leq i \leq m$, one can see that

$$|Y_m| = \prod_{i=1}^{m}(m+i) \cdot \left| \begin{array}{c|c} 1 & 0 \\ \hline 1 & T_m \end{array} \right| = \frac{(2m)!}{m!} \cdot |T_m| = (-1)^k \frac{(2m)!}{m!} \cdot |H_m|,$$

where T_m is the $m \times m$ Toeplitz matrix with $t_i = 1/(m+i)$ for $0 \leq |i| \leq m-1$, $k = m/2$ if m is even and $k = (m-1)/2$ if m is odd. We note that the last equality comes by changing i-th row with $(m-i+1)$-th row of T_m for $1 \leq j \leq k$, with k as above.

4 Proof of Theorem 1.1

Assume that $n \geq 2$ is an integer and $a_0, a_1, \cdots, a_n$ are distinct nonzero elements of $\mathbb{F}$. We let $d(\ell, e) = a_\ell - a_e$ for indexes $0 \leq \ell \neq e \leq n$. Let r and m be integers satisfying $1 \leq m \leq r+1 \leq n-r$ and consider the following $m \times m$ sub-matrix

$$C_m = \begin{bmatrix} C_{(i_1-r,e_1)} & C_{(i_1-r,e_2)} & \cdots & C_{(i_1-r,e_m)} \\ C_{(i_2-r,e_1)} & C_{(i_2-r,e_2)} & \cdots & C_{(i_2-r,e_m)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{(i_m-r,e_1)} & C_{(i_m-r,e_2)} & \cdots & C_{(i_m-r,e_m)} \end{bmatrix},$$

where $r+1 \leq i_1, \ldots, i_m \leq n$ and $0 \leq e_1, \cdots, e_m \leq r$, of the matrix $C = [C_{(i-r,j)}]$ defined in Section 1. To prove the part (i) of Theorem 1.1, we demonstrate the following two cases:

Case 1: If $0 = e_1 < e_2 < \cdots < e_m \leq r$ then C_m has nonzero determinant given by

$$\begin{aligned} |C_m| = (-1)^{m(r+1)+e_2+\cdots+e_m-1} \cdot \left(\prod_{\ell \in I} a_\ell \right)^{m-1} \cdot \prod_{k<\ell \in I_0} d(\ell,k) \cdot \prod_{j=2}^{m} \left(\prod_{k<\ell \in I_{e_j}} d(\ell,k) \right) \\ \cdot \left(\prod_{s=1}^{m} \prod_{\ell \in I \setminus \{e_2,\cdots,e_m\}} d(i_s,\ell) \right) \cdot \prod_{2 \leq k<\ell \leq m} d(i_k,i_\ell) \cdot \prod_{1 \leq s'<s \leq m} d(i_s,i_{s'}). \end{aligned} \tag{5}$$

By the equation (1), for $1 \leq s \leq m$ we have:

$$\begin{aligned} C_{(i_s-r,0)} &= (-1)^r \prod_{\ell \in I} d(i_s,\ell) \cdot \prod_{k<\ell \in I_0} d(\ell,k) \\ C_{(i_s-r,e_j)} &= (-1)^{r+e_j}\, a_{i_s} \cdot \prod_{\ell \in I_{e_j}} a_\ell \cdot \prod_{\ell \in I_{e_j}} d(i_s,\ell) \cdot \prod_{k<\ell \in I_{e_j}} d(\ell,k) \end{aligned}$$

Extracting $(-1)^r \prod_{k<\ell \in I_0} d(\ell,k)$ from the first column and factoring the term

$$(-1)^{r+e_j} \prod_{\ell \in I_{e_j}} a_\ell \cdot \prod_{k<\ell \in I_{e_j}} d(\ell,k)$$

from the e_j-th column $(2 \leq j \leq m)$ of the matrix C_m, one obtains that

$$|C_m| = (-1)^{mr+e_2+\cdots+e_m} \prod_{k<\ell \in I_0} d(\ell,k) \cdot \prod_{j=2}^{m} \left(\prod_{\ell \in I_{e_j}} a_\ell \cdot \prod_{k<\ell \in I_{e_j}} d(\ell,k) \right) \cdot |C'_m|, \tag{6}$$

where C'_m is the following $m \times m$ matrix

$$C'_m = \begin{bmatrix} \prod_{\ell \in I} d(i_1,\ell) & a_{i_1} \prod_{\ell \in I_{e_2}} d(i_1,\ell) & \cdots & a_{i_1} \prod_{\ell \in I_{e_m}} d(i_1,\ell) \\ \prod_{\ell \in I} d(i_2,\ell) & a_{i_2} \prod_{\ell \in I_{e_2}} d(i_2,\ell) & \cdots & a_{i_2} \prod_{\ell \in I_{e_m}} d(i_2,\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{\ell \in I} d(i_m,\ell) & a_{i_m} \prod_{\ell \in I_{e_2}} d(i_m,\ell) & \cdots & a_{i_m} \prod_{\ell \in I_{e_m}} d(i_m,\ell) \end{bmatrix}.$$

Then, dividing the s-th row by $\prod_{\ell\in I} d_{(i_s,\ell)}$ for $1 \leq s \leq m$, we obtain

$$|C'_m| = \prod_{s=1}^{m}\prod_{\ell\in I} d(i_s,\ell) \cdot \begin{vmatrix} 1 & \frac{a_{i_1}}{d(i_1,e_2)} & \cdots & \frac{a_{i_1}}{d(i_1,e_m)} \\ 1 & \frac{a_{i_2}}{d(i_2,e_2)} & \cdots & \frac{a_{i_2}}{d(i_2,e_m)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{a_{i_m}}{d(i_m,e_2)} & \cdots & \frac{a_{i_m}}{d(i_m,e_m)} \end{vmatrix}.$$

By Proposition 3.1, the last determinant is equal to $|Y_{m-1}|$ defined by the sets $\{a_{i_1}, \cdots, a_{i_m}\}$ and $\{a_{e_2}, \cdots, a_{e_m}\}$. Thus,

$$|C'_m| = \prod_{s=1}^{m}\prod_{\ell\in I} d(i_s,\ell) \cdot \frac{(-1)^{m-1}\prod_{j=2}^{m} a_{e_j} \cdot \prod_{2\leq k<\ell\leq m} d(i_k,i_\ell) \cdot \prod_{1\leq s'<s\leq m} d(i_s,i_{s'})}{\prod_{s=1}^{m}\prod_{\ell=2}^{m} d(i_s,e_\ell)} \tag{7}$$

Finally, putting (7) and (6) together, doing some algebraic simplification, one can get the desired equation (5), which is nonzero, since all of a_i's are distinct.

Case 2: If $1 \leq e_1 < e_2 < \cdots < e_m \leq r$, then C_m has nonzero determinant given by

$$\begin{aligned} |C_m| = (-1)^{mr+e_2+\cdots+e_m} \cdot \prod_{s=1}^{m} a_{i_s} \cdot \prod_{j=1}^{m}\left(\prod_{\ell\in I_{e_j}} a_\ell \cdot \prod_{k<\ell\in I_{e_j}} d(\ell,k)\right) \\ \cdot \left(\prod_{s=1}^{m}\prod_{\ell\in I\setminus\{e_1,\cdots,e_m\}} d(i_s,\ell)\right) \cdot \prod_{1\leq k<\ell\leq m} d(e_k,e_\ell) \cdot \prod_{1\leq s'<s\leq m} d(i_s,i_{s'}). \end{aligned} \tag{8}$$

Extracting $(-1)^{r+e_j} \cdot \prod_{\ell\in I_{e_j}} a_\ell \cdot \prod_{k<\ell\in I_{e_j}} d(\ell,k)$ from e_j-th column and $a_{i_s} \cdot \prod_{\ell\in I} d(i_s,\ell)$ from s-th row of C_m, for $1 \leq s,j \leq m$, lead to the following

$$|C_m| = (-1)^{mr+e_1+\cdots+e_m} \prod_{j=1}^{m}\left(\prod_{\ell\in I_{e_j}} a_\ell \cdot \prod_{k<\ell\in I_{e_j}} d(\ell,k)\right) \cdot \prod_{s=1}^{m}\left(a_{i_s} \cdot \prod_{\ell\in I} d(i_s,\ell)\right) \cdot |X_m|$$

where X_m is the Cauchy matrix defined by $\{a_{i_1}, \cdots, a_{i_m}\}$ and $\{a_{e_1}, \ldots, a_{e_m}\}$. By using the Cauchy's determinant formula (2), and doing some simplifications one can get the desired formula 8 for $|C_m|$, which is clearly nonzero. Thus, we have completed the proof of part (i) of Theorem 1.1.

Finally, in order to prove the part (ii), we recall that $\mathbf{C}_r^n = [C|D]$ is an $(n-r)\times(n+1)$ blocked matrix, for the integers $2 \leq r < n$, where $C = [C_{(i-r,j)}]$ is $(n-r)\times(r+1)$ and D is an $(n-r)\times(n-r)$ diagonal matrix with entries D_r. By part (i) of the main theorem, any $m \times m$ sub-matrix of the matrix C has nonzero determinant and hence C has maximal rank equal to $\min\{n-r, r+1\}$. It is clear that the matrix D has full rank equal to $n-r$. By exchanging the columns, if it is necessary, one can see that any $(n-r)\times(n-r)$ sub-matrix of $\mathbf{C}_r^n$ is a diagonal blocked matrix with blocks equal to D_r or $m \times m$ sub-matrices of C with $1 \leq m \leq \min\{n-r, r+1\}$, which have non-zero determinant. Therefore, any $(n-r)\times(n-r)$ sub-matrix of $\mathbf{C}_r^n$ has nonzero determinant, and hence it has maximal rank $n-r$, as desired.

5 A numerical example

Here, we provide a numerical example for the main theorem over $\mathbb{R}$ the field of real numbers and $\mathbb{F}_{11}$ the finite field with 11 elements. We take $r = 3$, $n = 7$ and $a_i = i + 1$ for $0 \leq i \leq 7$. Then, we have $D_3 = -288 \equiv 9 \pmod{11}$ and

$$\mathbf{C}_3^7 = \begin{bmatrix} -48 & 2880 & -2880 & 1440 & -288 & 0 & 0 & 0 \\ -240 & 12960 & -11520 & 4320 & 0 & -288 & 0 & 0 \\ -720 & 36288 & -30240 & 10080 & 0 & 0 & -288 & 0 \\ -1680 & 80640 & -64512 & 20160 & 0 & 0 & 0 & -288 \end{bmatrix}$$
$$\equiv \begin{bmatrix} 7 & 9 & 2 & 10 & 9 & 0 & 0 & 0 \\ 2 & 2 & 8 & 8 & 0 & 9 & 0 & 0 \\ 6 & 10 & 10 & 4 & 0 & 0 & 9 & 0 \\ 3 & 10 & 3 & 8 & 0 & 0 & 0 & 9 \end{bmatrix} (\bmod 11)$$

We note that the rank of above matrix is 4 as desired and the sub-matrix

$$C = \begin{bmatrix} -48 & 2880 & -2880 & 1440 \\ -240 & 12960 & -11520 & 4320 \\ -720 & 36288 & -30240 & 10080 \\ -1680 & 80640 & -64512 & 20160 \end{bmatrix} \equiv \begin{bmatrix} 7 & 9 & 2 & 10 \\ 2 & 2 & 8 & 8 \\ 6 & 10 & 10 & 4 \\ 3 & 10 & 3 & 8 \end{bmatrix} (\bmod 11),$$

has determinant $|C| = 40131624960 \equiv 9 \pmod{11}$. One can easily check that its 2×2 and 3×3 sub-matrices have nonzero determinants as desired. We leave the details for the interested readers.

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References

[1] CAUCHY, A. L.: **Mémorie sur les fonctions alternées et sur les somme alternées**. Exercises d' Analyse et de Phys. Math., II: 151-159, 1841.

[2] DAVIS, P. J.: **Interpolation and approximation**. Dover Publication Inc., New-York, 1975.

[3] ENGLIS, M.: **Toeplitz operators and group representations**. Journal of Fourier Analysis and Applications, 13:243-265, 2007.

[4] EULER, R.: **Characterizing bipartite Toeplitz graphs**. Theoretical Computer Science, 263:47-58, 2001.

[5] HILBERT, D.: **Ein betrag zur theoreie des Legendre'schen polynoms**. Acta Mathematica, 18:155-159, 1894.

[6] LACAN, J.; FIMES, J.: **A Construction of Matrices with No Singular Square Submatrices** International Conference on Finite Fields and Applications, Fq 2003, Finite Fields and Applications, Lecture Notes in Computer Science, vol 2948:145-147, 2004.

[7] PÓLYA, G.; SZEGO, G.: **Problems and Theorems in Analysis II**, Springer, Berlin, 1976.

[8] RIETSCH, K.: **Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties**. Journal of American Mathematical Society, 16:363-392, 2003.

[9] SCECHTER, S.: **On inversion of certain matrices.** Mathematical tables an other Aids to computations, 13: 76-77, 1959.

[10] SLOANE, N. J. A.: **The on-line encyclopedia of integer sequences.** http://www.research.att.com/~njas/sequences

[11] TOEPLITZ, O.: **Zur Transformation der Scharen bilinearer Formen von unendlichvielen Veränderlichen,** Nachr. der Kgl. Gessellschaft der Wissenschaften zu Göttingen, Mathematisch Physikalische Klasse, 110-115, 1907.

[12] TOEPLITZ, O.: **Zur Theorie der quadratischen und bilinearen Formen von unendlichvielen Veränderlichen. I. Teil:** Theorie der L-Formen, Math. Ann. 70:351-376, 1911.