# A survey on modular vector fields and CY modular forms attached to Dwork family

*Uma revisão de campos vetoriais modulares e formas modulares de CY decorrentes da família de Dwork*

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## Resumo

Este artigo tem como objetivo fornecer uma revisão geral dos trabalhos do autor sobre campos vetoriais modulares e formas modulares de Calabi-Yau (CY) decorrentes da família de Dwork. Para qualquer número inteiro positivo $n$, é introduzido um espaço modui $\mathsf{T} := \mathsf{T}_n$ de variedades enriquecidas de CY de dimensão $n$ provenientes da família de Dwork. Observa-se que existe um único campo vetorial $\mathsf{D}$ em $\mathsf{T}$, conhecido como campo vetorial modular, cujos componentes de uma solução podem ser expressos como $q$-expansões (série de Fourier) com coeficientes inteiros. Chamamos essas $q$-expansões de formas modulares de CY e verifica-se que o espaço gerado por elas possui uma estrutura canônica de $\mathfrak{sl}_2(\mathbb{C})$-módulo que lhe confere uma estrutura algébrica de Rankin-Cohen. Todos esses conceitos são explicitamente estabelecidos para $n = 1, 2, 3, 4$.

**Palavras-chave:** Campo vetorial modular, forma modular de Calabi-Yau, conexão de Gauss-Manin disfarçada.

## Abstract

This article aims to give a survey of the works of the author on modular vector fields and Calabi-Yau (CY) modular forms attached to the Dwork family and avoid technical details. For any positive integer $n$, it is introduced a moduli space $\mathsf{T} := \mathsf{T}_n$ of enhanced CY $n$-folds arising from the Dwork family. It is observed that there exists a unique vector field $\mathsf{D}$ in $\mathsf{T}$, known as modular vector field, whose solution components can be expressed as $q$-expansions

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(Fourier series) with integer coefficients. We call these $q$-expansions CY modular forms and it is verified that the space generated by them has a canonical $\mathfrak{sl}_2(\mathbb{C})$-module structure which provides it with a Rankin-Cohen algebraic structure. All these concepts are explicitly established for $n = 1, 2, 3, 4$.

**Keywords:** Modular vector field, Calabi-Yau modular form, Gauss-Manin conection in disguise.

**MSC2010: 11F11, 32M25, 34M45, 14J32, 14J15, 14N35.**

# 1    Introduction

Since introducing Calabi-Yau varieties, a vast number of works in mathematics and theoretical physics have been dedicated to the study of related differential equations. The solutions of these differential equations, or system of differential equations, provide us with innumerous infinite series or $q$-expansions (Fourier series) with integer coefficients which are generating functions of certain quantities. In lower dimensions $n = 1$ and $n = 2$ which are related to the elliptic curves and K3 surfaces, usually these q-expansions are (quasi-)modular forms, however, in higher dimensions we can not relate them with classical quasi-modular forms. Hossein Movasati by using an algebraic method in a geometric framework, calling *Gauss-Manin connection in disguise* (GMCD), introduced in a more systematic way a finite number of certain $q$-expansions arising from a family of CY varieties which conjecturally can generate all other $q$-expansions emerged from the same family. He called these finite number of $q$-expansions as *CY modular forms*. Indeed, CY modular forms are solution components of a unique canonical vector field, calling *modular vector field*, in a moduli space of the considered family of CY varieties enhanced with a certain basis of the middle de Rham cohomology space. To understand better the GMCD one can start reading the paper [5] which applies the method to the families of elliptic curves, and then continue with the paper [6] or the book [7]. The author in a joint work with Movasati [8] applied GMCD to a family of CY $n$-folds, $n \in \mathbb{N}$, arising from the Dwork family, and then pushed the studies forward in subsequent papers [9, 10, 11]. The present article gives a survey of [8, 9, 10, 11] and states explicitly the essential ingredients and objects in dimensions $n = 1, 2, 3, 4$.

The present article is prepared as follows. In Section 2 we first construct a moduli space arising from the Dwork family, and then establish and discuss the main results of [8, 9, 10] for any dimension $n$. We state explicitly the modular vector field, the associated $\mathfrak{sl}_2(\mathbb{C})$-module structure, solution components and some other related

facts in dimensions $n = 1, 2, 3, 4$, respectively, in Section 3, Section 4, Section 5, Section 6.

# 2 GMCD, $\mathfrak{sl}_2(\mathbb{C})$-mdodule structure, CY modular forms and algebraic RC structure

For any positive integer $n$, similarly to mirror quintic family, we construct a one-parameter family $X := X_z$, $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, of Calabi-Yau $n$-folds arising from the Dwork family:

$$W_z := \{zx_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)x_0x_1x_2\cdots x_{n+1} = 0\} \subset \mathbb{P}^{n+1},$$

and then obtain the moduli space $\mathsf{T} := \mathsf{T}_n$ of the enhanced pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ is a basis of the $n$-th algebraic de Rham cohomology $H_{\mathrm{dR}}^n(X)$ satisfying some specific properties. Indeed, we find:

$$\mathsf{T} = \mathrm{Spec}\left(\mathbb{C}\Big[t_1, t_2, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}\Big]\right), \tag{2.1}$$

$$\mathscr{O}_{\mathsf{T}} = \mathbb{C}\Big[t_1, t_2, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}\Big], \tag{2.2}$$

in which $\check{t}$ is the product of $\left[\frac{n+1}{2}\right]$ number of $t_j$'s and

$$\mathsf{d} := \mathsf{d}_n = \dim \mathsf{T} = \begin{cases} \frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd;} \\[2mm] \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even.} \end{cases} \tag{2.3}$$

Moreover, $\Delta = t_{n+2}(t_{n+2} - t_1^{n+2})$ is the discriminant of the modified Dwork family under the transformation $z = \frac{t_{n+2}}{t_1^{n+2}}$. We observe that (see [8, Theorem 1.1]) there exist a unique vector field $\mathsf{D} := \mathsf{D}_n$ and unique regular functions $\mathsf{Y}_j \in \mathscr{O}_{\mathsf{T}}$, $j = 1, 2, \ldots, n-2$ in $\mathsf{T}$ such that the Gauss-Manin connection of the universal family of $\mathsf{T}$ composed with the vector field $\mathsf{D}$, namely $\nabla_{\mathsf{D}}$, satisfies (no worries if you do not

know Gauss-Manin connection, just keep reading and forget it):

$$\nabla_{\mathsf{D}}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\vdots\\\alpha_n\\\alpha_{n+1}\end{pmatrix}=\underbrace{\begin{pmatrix}0&1&0&0&\cdots&0&0\\0&0&\mathsf{Y}_1&0&\cdots&0&0\\0&0&0&\mathsf{Y}_2&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&0&\cdots&\mathsf{Y}_{n-2}&0\\0&0&0&0&\cdots&0&-1\\0&0&0&0&\cdots&0&0\end{pmatrix}}_{\mathsf{Y}}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\vdots\\\alpha_n\\\alpha_{n+1}\end{pmatrix}. \tag{2.4}$$

We conjecture that component solutions of $\mathsf{D}$ can be expressed as $q$-expansions with integer coefficients (this is verified for $n = 1, 2, 3, 4$, which are stated in the next sections). For all $n$, we can find vector fields $\mathsf{W} := \mathsf{W}_n$ and $\delta := \delta_n$ in $\mathsf{T}$ which along with $\mathsf{D}$ form a copy of $\mathfrak{sl}_2(\mathbb{C})$ (see [9, Theorem 1.4]), i.e.:

$$[\mathsf{D}, \delta] = \mathsf{W}, \quad [\mathsf{W}, \mathsf{D}] = 2\mathsf{D}, \quad [\mathsf{W}, \delta] = -2\delta,$$

where $[\cdot, \cdot]$ refers to the Lie bracket of vector fields. Note that the vector fields $\mathsf{D}, \mathsf{W}, \delta$ in [8, 9, 10, 11] were denoted by $\mathsf{R}, \mathsf{H}, \mathsf{F}$, respectively. Indeed, we observe that $\mathsf{W}$ and $\delta$ are in the following forms:

$$\mathsf{W} = \sum_{j=1}^{\mathsf{d}} w_j t_j \frac{\partial}{\partial t_j}, \text{ for some } w_j \in \mathbb{Z}_{\geq 0}, \tag{2.5}$$

$$\delta = \frac{\partial}{\partial t_2}, \text{ if } n \neq 2 \ (\delta = 2\frac{\partial}{\partial t_2}, \text{ if } n = 2). \tag{2.6}$$

We should mention that for all odd $n \geq 3$, as we will see in Section 5, we will need to use a simple transformation to get $\delta$ as above and substitute $t_{\mathsf{d}}$ by $\tilde{t}_{\mathsf{d}}$, which, by abuse of notation, will be denoted again by $t_{\mathsf{d}}$.

If alternately, by abuse of notation, we suppose that $t_j$'s are solution components of $\mathsf{D}$, then we can consider $\mathsf{D}, \mathsf{W}, \delta$ as differential operators on the $\mathbb{C}$-algebra generated by $t_j$'s :

$$\mathscr{M} := \mathbb{C}\Big[t_1, t_2, t_3, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\tilde{t}}\Big],$$

which is called the space of *CY modular forms*. By setting $\deg t_j := w_j$, we provide

the $\mathbb{C}$-algebra $\mathscr{M}$ with an algebraic graded structure, i.e.:

$$\mathscr{M} = \bigoplus_{k \in \mathbb{Z}} \mathscr{M}_k,$$

in which $\mathscr{M}_k := \{f \in \mathscr{M} \mid \deg f = k\}$, for all $k \in \mathbb{Z}$, is the space of CY modular forms of weight $k$. In particular, for all $n \in \mathbb{N}$, we observe that $w_1 = 1, w_2 = 2, w_{n+2} = n + 2$. Using the assigned degrees (weights) it turns out that $\mathsf{D}$ is a quasi-homogeneous vector field of degree 2 in $\mathsf{T}$, and consequently it is a degree 2 differential operator on $\mathscr{M}$, i.e., for all $f \in \mathscr{M}_k$, we get $\mathsf{D}f \in \mathscr{M}_{k+2}$. Analogously, we observe that $\mathsf{W}f = kf$, which is the operator multiplication by weight and $\delta f \in \mathscr{M}_{k-2}$ decreases the weight by 2.

By comparing the $\mathfrak{sl}_2(\mathbb{C})$-module structure of the space of full quasi-modular forms with the $\mathfrak{sl}_2(\mathbb{C})$-module structure of the space of CY modular forms $\mathscr{M}$, it turns out that $t_2$ plays the role of the quasi-modular form $E_2$ (which is the weight 2 Eisenstein series). In this way, we introduce the space of *2CY modular forms* $\mathscr{M}^2$ as follows:

$$\mathscr{M}^2 := \mathbb{C}\left[t_1, t_3, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}\right] = \bigoplus_{k \in \mathbb{Z}} \mathscr{M}^2_k.$$

Indeed, we have $\mathscr{M} = \mathscr{M}^2[t_2]$. In a conceptual comparison, the space of 2CY modular forms $\mathscr{M}^2$ is equivalent to the space of full modular forms $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$, and the space of CY modular forms $\mathscr{M} = \mathscr{M}^2[t_2]$ is equivalent to the space of full quasi-modular forms $\widetilde{\mathcal{M}}(\mathrm{SL}_2(\mathbb{Z})) = \mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))[E_2] = \mathbb{C}[E_2, E_4, E_6]$. The reason for choosing the name "2CY modular forms" is because of the order of appearance of this space in the literature. In fact the space of 1CY modular forms is $\mathscr{M}^1 := \mathbb{C}\left[t_1, t_{n+2}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})}\right]$ which has been studied by Movasati [7].

Let $f \in \mathscr{M}^2_k$ be a 2CY modular form of weight $k$ for some $k \in \mathbb{Z}$. It is obvious that $\mathsf{D}f \in \mathscr{M}_{k+2}$, however it is not necessarily a 2CY modular form. Analogous to the Ramanujan-Serre derivation for full modular forms, we define the degree 2 *Ramanujan-Serre-type* derivation $\partial : \mathscr{M}^2_* \to \mathscr{M}^2_{*+2}$ as follows:

$$\partial f = \mathsf{D}f + kt_2 f, \quad \text{if } n \neq 2, \tag{2.7}$$

$$\partial f = \mathsf{D}f + \frac{k}{2}t_2 f, \quad \text{if } n = 2. \tag{2.8}$$

Since $\mathsf{D}$ is a degree 2 derivation on $\mathscr{M}$, due to Zagier [12], we can provide $\mathscr{M}$ with a standard Rankin-Cohen (RC) structure by defining the *m-th RC bracket for*

*CY modular forms* as follows:

$$[f,g]_{\mathsf{D},m} := \sum_{i+j=m} (-1)^j \binom{m+k-1}{i}\binom{m+l-1}{j} \mathsf{D}^j f \mathsf{D}^i g, \ \forall f \in \mathscr{M}_k, \ \forall g \in \mathscr{M}_l, \tag{2.9}$$

where $m \in \mathbb{Z}_{\geq 0}$, $k, l \in \mathbb{Z}$ and $\mathsf{D}^j f, \mathsf{D}^i g$ are respectively the $j$-th and $i$-th derivative of $f$ and $g$ with respect to the derivation $\mathsf{D}$. It is evident that $[f,g]_{\mathsf{D},m} \in \mathscr{M}_{k+l+2m}$. Cohen proved that the RC bracket of modular forms is again a modular form. Similarly, we can observe that RC bracket of 2CY modular forms is again a 2CY modular form, i.e.,

$$\forall f \in \mathscr{M}^2_k, \ \forall g \in \mathscr{M}^2_l \Longrightarrow [f,g]_{\mathsf{D},m} \in \mathscr{M}^2_{k+l+2m}. \tag{2.10}$$

To prove this, we first let:

$$\Lambda := \begin{cases} -\frac{1}{2}\mathsf{D}t_2 - \frac{1}{4}t_2^2, & if \ n = 2, \\ -\mathsf{D}t_2 - t_2^2, & if \ n \neq 2, \end{cases} \tag{2.11}$$

and observe that $\Lambda \in \mathscr{M}^2_4$. For any $m \in \mathbb{Z}_{\geq 0}$ we define the brackets $[\cdot,\cdot]_{\partial,\Lambda,m} : \mathscr{M}^2_k \times \mathscr{M}^2_l \to \mathscr{M}^2_{k+l+2m}$:

$$[f,g]_{\partial,\Lambda,m} = \sum_{i+j=m} (-1)^j \binom{m+k-1}{i}\binom{m+l-1}{j} f_{(j)} g_{(i)}, \tag{2.12}$$

where $f \in \mathscr{M}^2_k$, $g \in \mathscr{M}^2_l$, and $f_{(j)} \in \mathscr{M}^2_{k+2j}$, $g_{(i)} \in \mathscr{M}^2_{l+2i}$ are defined recursively as follows

$$f_{(j+1)} = \partial f_{(j)} + j(j+k-1)\Lambda f_{(j-1)}, \ g_{(i+1)} = \partial g_{(i)} + i(i+l-1)\Lambda g_{(i-1)}, \tag{2.13}$$

with initial conditions $f_{(0)} = f$, $g_{(0)} = g$, $f_{(1)} = \partial f$, $g_{(0)} = \partial g$. Then we obtain:

$$[f,g]_{\mathsf{D},m} = [f,g]_{\partial,\Lambda,m},$$

which shows $[f,g]_{\mathsf{D},m} \in \mathscr{M}^2_{k+l+2m}$. By this we provide $(\mathscr{M}^2, [f,g]_{\mathsf{D},m})$ with a *canonical RC algebra structure*, in the sense Zagier [12].

In the subsequent sections we state $\mathsf{D}, \mathsf{W}, \delta$ and component solutions of $\mathsf{D}$ for $n = 1, 2, 3, 4$. In what follows $E_{2j}$, $j = 1, 2, 3$, are Eisenstein series defined as $E_{2j}(q) = 1 + b_j \sum_{k=1}^{\infty} \sigma_{2j-1}(k) q^k$ with $(b_1, b_2, b_3) = (-24, 240, -504)$ and $\sigma_j(k) = \sum_{d|k} d^j$, and $\eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k)$ is the Dedekind eta function. We also consider

$q = e^{2\pi i \tau}$ where $\tau \in \mathbb{C}$ with $\operatorname{Im} \tau > 0$.

# 3    The case $n = 1$

In this case we find:

$$\mathsf{D} = (-9(t_1^3 - t_3) - t_2 t_1)\frac{\partial}{\partial t_1} + (81 t_1(t_1^3 - t_3) - t_2^2)\frac{\partial}{\partial t_2} + (-3 t_2 t_3)\frac{\partial}{\partial t_3}, \qquad (3.1)$$

$$\mathsf{W} = t_1 \frac{\partial}{\partial t_1} + 2 t_2 \frac{\partial}{\partial t_2} + 3 t_3 \frac{\partial}{\partial t_3}, \qquad (3.2)$$

$$\delta = \frac{\partial}{\partial t_2}. \qquad (3.3)$$

In particular, vector field (3.2) implies $\deg(t_1) = 1$, $\deg(t_2) = 2$ and $\deg(t_3) = 3$. We find a solution of $\mathsf{D}$ in terms of quasi-modular forms in $\widetilde{\mathcal{M}}(\Gamma_0(3), \chi_{-3})$ (actually, $t_1$ and $t_3$ are modular forms and $t_2$ is a quasi-modular form, see [11, §3]). Indeed, by using the transformations $\mathcal{P}_3 = -2t_2 - 9t_1^2$, $\mathcal{Q}_3 = 9t_1^2$, $\mathcal{R}_3 = 3t_1 t_3$ and $\mathcal{S}_3 = t_3^2$, we find the following Ramanujan-type system for $\Gamma_0(3)$:

$$\begin{cases} \mathcal{P}_3' = \frac{1}{6}\big(\mathcal{P}_3^2 - \mathcal{Q}_3^2\big) \\ \mathcal{Q}_3' = \frac{1}{3}\big(\mathcal{P}_3 \mathcal{Q}_3 - \mathcal{Q}_3^2 + 54\mathcal{R}_3\big) \\ \mathcal{R}_3' = \frac{2}{3}\mathcal{P}_3 \mathcal{R}_3 + \frac{1}{3}\mathcal{Q}_3 \mathcal{R}_3 + 9\mathcal{S}_3 \\ \mathcal{S}_3' = \mathcal{P}_3 \mathcal{S}_3 + \mathcal{Q}_3 \mathcal{S}_3 \end{cases}, \quad *' = q\frac{\partial}{\partial q} = \frac{1}{2\pi i}\frac{d}{d\tau}, \qquad (3.4)$$

in which the relation $\mathcal{R}_3^2 - \mathcal{Q}_3 \mathcal{S}_3 = 0$ holds. A particular solution of this system is given as follows (see [11, Theorem 1.2]):

$$\begin{cases} \mathcal{P}_3(q) = \frac{1}{4}\big(E_2(q) + 3E_2(q^3)\big), \\ \mathcal{Q}_3(q) = \frac{1}{2}\big(3E_2(q^3) - E_2(q)\big), \\ \mathcal{R}_3(q) = \eta^8(q^3) + 9\frac{\eta^8(q^3)\eta^3(q^9)}{\eta^3(q)}, \\ \mathcal{S}_3(q) = \left(\frac{\eta^9(q^3)}{\eta^3(q)}\right)^2, \end{cases} \qquad (3.5)$$

where $\mathcal{Q}_3 \in \mathcal{M}_2(\Gamma_0(3))$, $\mathcal{R}_3 \in \mathcal{M}_4(\Gamma_0(3))$, $\mathcal{S}_3 \in \mathcal{M}_6(\Gamma_0(3))$ and $\mathcal{P}_3 \in \widetilde{\mathcal{M}}_2(\Gamma_0(3))$. Moreover, if we consider

$$\Delta_3 := \eta^6(q)\eta^6(q^3),$$

which is a cusp form of weight 6 for $\Gamma_0(3)$, then $\Delta_3 = \mathcal{Q}_3 \mathcal{R}_3 - 27\mathcal{S}_3$ and it is a factor of the discriminant of the Dwork family (which can be called modular discriminant for $\Gamma_0(3)$) satisfying:

$$\Delta_3' = \mathcal{P}_3 \Delta_3.$$

If we consider $\mathcal{Q}_3, \mathcal{R}_3, \mathcal{S}_3$ as free parameters and let $\mathcal{I} := \langle \mathcal{R}_3^2 - \mathcal{Q}_3\mathcal{S}_3 \rangle$ to be the ideal generated by $\mathcal{R}_3^2 - \mathcal{Q}_3\mathcal{S}_3$ in $\mathbb{C}[\mathcal{Q}_3, \mathcal{R}_3, \mathcal{S}_3]$, then:

$$\mathcal{M}(\Gamma_0(3)) \simeq \frac{\mathbb{C}[\mathcal{Q}_3, \mathcal{R}_3, \mathcal{S}_3]}{\mathcal{I}}, \tag{3.6}$$

$$\widetilde{\mathcal{M}}(\Gamma_0(3)) \simeq \frac{\mathbb{C}[\mathcal{P}_3, \mathcal{Q}_3, \mathcal{R}_3, \mathcal{S}_3]}{\mathcal{I}}. \tag{3.7}$$

# 4    The case $n = 2$

In this case we find:

$$\mathsf{D} = (t_3 - t_2 t_1)\frac{\partial}{\partial t_1} + \left(2t_1^2 - \frac{1}{2}t_2^2\right)\frac{\partial}{\partial t_2} + (8t_1^3 - 2t_2 t_3)\frac{\partial}{\partial t_3} + (-4t_2 t_4)\frac{\partial}{\partial t_4}, \tag{4.1}$$

$$\mathsf{W} = 2t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 4t_3\frac{\partial}{\partial t_3} + 8t_4\frac{\partial}{\partial t_4}, \tag{4.2}$$

$$\delta = 2\frac{\partial}{\partial t_2}, \tag{4.3}$$

where the polynomial equation $t_3^2 = 4(t_1^4 - t_4)$ holds among $t_i$'s. From (4.2) we get $\deg(t_1) = 2$, $\deg(t_2) = 2$, $\deg(t_3) = 4$ and $\deg(t_4) = 8$. We get a solution of $\mathsf{D}$ in terms of quasi-modular forms in $\widetilde{\mathcal{M}}(\Gamma_0(2))$ (actually, $t_1$ and $t_3$ are modular forms and $t_2$ is a quasi-modular form). Indeed, by using the transformations $\mathcal{P}_2 = 20t_2$, $\mathcal{Q}_2 = 40t_1$ and $\mathcal{R}_2 = 800t_3$, we find the following Ramanujan-type system for $\Gamma_0(2)$:

$$\begin{cases} \mathcal{P}_2' = \frac{1}{8}(\mathcal{P}_2^2 - \mathcal{Q}_2^2) \\ \mathcal{Q}_2' = \frac{1}{4}(\mathcal{P}_2\mathcal{Q}_2 - \mathcal{R}_2) \quad , \qquad *' = q\frac{\partial *}{\partial q}, \\ \mathcal{R}_2' = \frac{1}{2}(\mathcal{P}_2\mathcal{R}_2 - \mathcal{Q}_2^3) \end{cases} \tag{4.4}$$

whose a particular solution is given as follows (see [11, Theorem 1.1]):

$$\begin{cases} \mathcal{P}_2(q) = \frac{1}{3}(E_2(q) + 2E_2(q^2)), \\ \mathcal{Q}_2(q) = 2E_2(q^2) - E_2(q), \\ \mathcal{R}_2(q) = \frac{1}{3}(4E_4(q^2) - E_4(q)), \end{cases} \tag{4.5}$$

in which $\mathcal{Q}_2 \in \mathcal{M}_2(\Gamma_0(2))$, $\mathcal{R}_2 \in \mathcal{M}_4(\Gamma_0(2))$ and $\mathcal{P}_2 \in \widetilde{\mathcal{M}}_2(\Gamma_0(2))$. Moreover, if we consider

$$\Delta_2 := \eta^8(q)\eta^8(q^2),$$

which is a cusp form of weight 8 for $\Gamma_0(2)$, then $\Delta_2 = \frac{1}{256}(\mathcal{Q}_2^4 - \mathcal{R}_2^2)$ and it is a factor of the discriminant of the Dwork family (which can be called modular discriminant

for $\Gamma_0(2)$) satisfying:

$$\Delta_2' = \mathcal{P}_2\Delta_2.$$

We also have:

$$\mathcal{M}(\Gamma_0(2)) = \mathbb{C}[\mathcal{Q}_2, \mathcal{R}_2] \tag{4.6}$$

$$\widetilde{\mathcal{M}}(\Gamma_0(2)) = \mathbb{C}[\mathcal{P}_2, \mathcal{Q}_2, \mathcal{R}_2] \tag{4.7}$$

# 5   The case $n = 3$

In this case we obtain:

$$\mathsf{D} = \left(t_3 - t_2 t_1\right)\frac{\partial}{\partial t_1} + \left(\frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - t_2^2\right)\frac{\partial}{\partial t_2} \tag{5.1}$$

$$+ \left(\frac{t_3^3 t_6}{5^4(t_1^5 - t_5)} - 3t_2 t_3\right)\frac{\partial}{\partial t_3} + \left(-t_7 - t_2 t_4\right)\frac{\partial}{\partial t_4}$$

$$+ \left(-5t_2 t_5\right)\frac{\partial}{\partial t_5} + \left(5^5 t_1^3 - 2t_3 t_4 - t_2 t_6\right)\frac{\partial}{\partial t_6} + \left(-5^4 t_1 t_3 - t_2 t_7\right)\frac{\partial}{\partial t_7},$$

$$\mathsf{W} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 5t_5\frac{\partial}{\partial t_5} + t_6\frac{\partial}{\partial t_6} + 2t_7\frac{\partial}{\partial t_7}, \tag{5.2}$$

$$\delta = \frac{\partial}{\partial t_2} - t_4\frac{\partial}{\partial t_7}. \tag{5.3}$$

Hence $\deg(t_1) = 1$, $\deg(t_2) = 2$, $\deg(t_3) = 3$, $\deg(t_4) = 0$, $\deg(t_5) = 5$, $\deg(t_6) = 1$, $\deg(t_7) = 2$. We can find the $q$-expansion of a solution of $\mathsf{D}$, whose first 7 coefficients are given in Table 1.

| | $q^0$ | $q^1$ | $q^2$ | $q^3$ | $q^4$ | $q^5$ | $q^6$ |
|---|---|---|---|---|---|---|---|
| $\frac{1}{24}t_1$ | $\frac{1}{120}$ | 1 | 175 | 117625 | 111784375 | 126958105626 | 160715581780591 |
| $\frac{1}{10}t_2$ | $-\frac{1}{10}$ | 17 | 11185 | 12261425 | 16166719625 | 23478405649152 | 36191848368238417 |
| $\frac{1}{10}t_3$ | $-\frac{1}{50}$ | 13 | 6425 | 6744325 | 8719953625 | 12525150549888 | 19171976431076873 |
| $\frac{1}{125}t_4$ | $-\frac{1}{5}$ | 13 | 2860 | 1855775 | 1750773750 | 1981335668498 | 2502724752660128 |
| $-t_5$ | 0 | -1 | 170 | 41475 | 32183000 | 32678171250 | 38612049889554 |
| $\frac{1}{25}t_6$ | $-\frac{3}{5}$ | 187 | 28760 | 16677425 | 15028305250 | 16597280453022 | 20644227272244012 |
| $\frac{-1}{750}t_7$ | $\frac{1}{30}$ | 3 | 930 | 566375 | 526770000 | 592132503858 | 745012928951258 |

Table 1: Coefficients of $q^k$, $0 \le k \le 6$, in the q-expansion of a solution of $\mathsf{D} = \mathsf{D}_3$.

Using this solution we obtain:

$$\mathsf{Y}_1 = \frac{t_3^3}{5^4(t_1^5 - t_5)} = 5 + 2875\frac{q}{1-q} + 609250 \times 2^3\frac{q^2}{1-q^2} + \dots$$

which is the Yukawa coupling given in [1].

As we can see, in this case $\delta$ is different from the form claimed in (2.6) (this

happens in all odd cases $\geq 3$). We can solve this problem using the transformation:

$$\tilde{t}_7 := t_7 + t_2 t_4.$$

from which we obtain:

$$\mathsf{D}(\tilde{t}_7) = -5^4 t_1 t_3 + \frac{t_3^3 t_4^2}{5^4(t_1^5 - t_5)} - 2 t_2 \tilde{t}_7,$$

$$\mathsf{W}(\tilde{t}_7) = 2 \tilde{t}_7,$$

$$\delta(\tilde{t}_7) = 0.$$

Hence we get the vector fields $\mathsf{D}$, $\mathsf{W}$ and $\delta$ in the chart $(t_1, t_2, \ldots, t_6, \tilde{t}_7)$ as follows:

$$\mathsf{D} = \left( t_3 - t_2 t_1 \right) \frac{\partial}{\partial t_1} + \left( \frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - t_2^2 \right) \frac{\partial}{\partial t_2} \tag{5.4}$$

$$+ \left( \frac{t_3^3 t_6}{5^4(t_1^5 - t_5)} - 3 t_2 t_3 \right) \frac{\partial}{\partial t_3} - \tilde{t}_7 \frac{\partial}{\partial t_4} - 5 t_2 t_5 \frac{\partial}{\partial t_5}$$

$$+ \left( 5^5 t_1^3 - 2 t_3 t_4 - t_2 t_6 \right) \frac{\partial}{\partial t_6} + \left( -5^4 t_1 t_3 + \frac{t_3^3 t_4^2}{5^4(t_1^5 - t_5)} - 2 t_2 \tilde{t}_7 \right) \frac{\partial}{\partial \tilde{t}_7},$$

$$\mathsf{W} = t_1 \frac{\partial}{\partial t_1} + 2 t_2 \frac{\partial}{\partial t_2} + 3 t_3 \frac{\partial}{\partial t_3} + 5 t_5 \frac{\partial}{\partial t_5} + t_6 \frac{\partial}{\partial t_6} + 2 \tilde{t}_7 \frac{\partial}{\partial \tilde{t}_7}, \tag{5.5}$$

$$\delta = \frac{\partial}{\partial t_2}. \tag{5.6}$$

Indeed, here $t_7 \notin \mathscr{M}^2{}_2$, hence the space of 2CY modular forms is the following:

$$\mathscr{M}^2 = \mathbb{C}\left[ t_1, t_3, t_4, t_5, t_6, \tilde{t}_7, \frac{1}{t_5(t_5 - t_1^5)} \right].$$

This transformation is new and it does not appear in pervious works.

Note that in [10] to solve the above problem we considered:

$$\tilde{\mathsf{D}} := \mathsf{D} - t_2 \left( [\mathsf{D}, \frac{\partial}{\partial t_2}] - \mathsf{W} \right) = \mathsf{D} + t_2 t_4 \frac{\partial}{\partial t_4} - t_2 t_7 \frac{\partial}{\partial t_7},$$

then we get:

$$\tilde{\mathsf{D}} = \left(t_3 - t_1 t_2\right)\frac{\partial}{\partial t_1} + \left(\frac{t_3^3 t_4}{5^4(t_1^5 - t_5)} - t_2^2\right)\frac{\partial}{\partial t_2} \tag{5.7}$$

$$+ \left(\frac{t_3^3 t_6}{5^4(t_1^5 - t_5)} - 3t_2 t_3\right)\frac{\partial}{\partial t_3} + \left(- t_7\right)\frac{\partial}{\partial t_4}$$

$$+ \left(- 5t_2 t_5\right)\frac{\partial}{\partial t_5} + \left(5^5 t_1^3 - 2t_3 t_4 - t_2 t_6\right)\frac{\partial}{\partial t_6} + \left(- 5^4 t_1 t_3 - 2t_2 t_7\right)\frac{\partial}{\partial t_7},$$

$$\tilde{\mathsf{W}} = \mathsf{W} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 5t_5\frac{\partial}{\partial t_5} + t_6\frac{\partial}{\partial t_6} + 2t_7\frac{\partial}{\partial t_7}, \tag{5.8}$$

$$\tilde{\delta} = \frac{\partial}{\partial t_2}, \tag{5.9}$$

which again form a copy of $\mathfrak{sl}_2(\mathbb{C})$. But we still could not find the $q$-expansion of a solution of $\tilde{\mathsf{D}}$, which is not interesting. The Gauss-Manin connection matrix of $\tilde{\mathsf{D}}$ is as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 \\ t_2 t_4 & 0 & 0 & -1 \\ -t_2(t_2 t_4 + t_7) & t_2 t_4 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

# 6   The case $n = 4$

In this case we get:

$$\mathsf{D} = \left(t_3 - t_2 t_1\right)\frac{\partial}{\partial t_1} + \left(\frac{6^{-2} t_3^2 t_4 t_8}{t_1^6 - t_6} - t_2^2\right)\frac{\partial}{\partial t_2} \tag{6.1}$$

$$+ \left(\frac{6^{-2} t_3^2 t_5 t_8}{t_1^6 - t_6} - 3t_2 t_3\right)\frac{\partial}{\partial t_3} + \left(\frac{-6^{-2} t_3^2 t_7 t_8}{t_1^6 - t_6} - t_2 t_4\right)\frac{\partial}{\partial t_4}$$

$$+ \left(\frac{6^{-2} t_3 t_5^2 t_8 + 5t_1^4 t_3 t_8}{2(t_1^6 - t_6)} - t_3 t_4 - 2t_2 t_5\right)\frac{\partial}{\partial t_5}$$

$$+ \left(- 6t_2 t_6\right)\frac{\partial}{\partial t_6} + \left(\frac{6^{-2} t_4^2 - t_1^2}{2 \times 6^{-2}}\right)\frac{\partial}{\partial t_7} + \left(\frac{3t_1^5 t_3 t_8}{t_1^6 - t_6} - 3t_2 t_8\right)\frac{\partial}{\partial t_8},$$

$$\mathsf{W} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + t_4\frac{\partial}{\partial t_4} + 2t_5\frac{\partial}{\partial t_5} + 6t_6\frac{\partial}{\partial t_6} + 3t_8\frac{\partial}{\partial t_8}, \tag{6.2}$$

$$\delta = \frac{\partial}{\partial t_2}, \tag{6.3}$$

where the equation $t_8^2 = 36(t_1^6 - t_6)$ holds among $t_i$'s. Analogous to the pervious cases we have $\deg(\mathsf{t}_1) = 1$, $\deg(\mathsf{t}_2) = 2$, $\deg(\mathsf{t}_3) = 3$, $\deg(\mathsf{t}_4) = 1$, $\deg(\mathsf{t}_5) = 2$, $\deg(\mathsf{t}_6) = 6$, $\deg(\mathsf{t}_7) = 0$, $\deg(\mathsf{t}_8) = 3$. In this case also we can find the $q$-expansion of a solution components of $\mathsf{D}$ and their first 7 coefficients are given in Table 2. If we

compute the $q$-expansion of $\mathsf{Y}_1^2$, then we find

$$\frac{1}{6}\mathsf{Y}_1^2 = \frac{1}{6}(-\mathsf{Y}_2)^2 = \frac{1}{6^3}\frac{t_3^4}{t_1^6 - t_6}$$

$$= 6 + 120960\,q + 4136832000\,q^2 + 148146924602880\,q^3 + 5420219848911544320\,q^4$$

$$+ 200623934537137119778560\,q^5 + \dots$$

which is the 4-point function discussed in [3, Table 1, $d = 4$]. We have also computed the $q$-expansion of the modular coordinate $z$

$$\frac{z}{6^6} = \frac{t_6}{(6t_1)^6} = q - 6264q^2 - 8627796q^3 - 237290958144q^4 - 4523787606611250q^5 + \cdots \tag{6.4}$$

which coincides with the one computed in [4, §6.1].

| | $q^0$ | $q^1$ | $q^2$ | $q^3$ | $q^4$ | $q^5$ | $q^6$ |
|---|---|---|---|---|---|---|---|
| $\frac{1}{20}t_1$ | $\frac{1}{720}$ | 1 | 4131 | 51734044 | 918902851011 | 19562918469120126 | 465569724397794578388 |
| $\frac{1}{216}t_2$ | $-\frac{1}{216}$ | 9 | 110703 | 2248267748 | 55181044614231 | 1498877559908208054 | 43378802521495632926652 |
| $\frac{1}{14}t_3$ | $-\frac{1}{504}$ | 11 | 115137 | 2265573692 | 54820079452449 | 1477052190387154386 | 42523861222488896739828 |
| $\frac{1}{24}t_4$ | $-\frac{1}{144}$ | 16 | 193131 | 3904146832 | 95619949713765 | 2594164605185043648 | 75018247757143686903060 |
| $\frac{1}{2}t_5$ | $-\frac{1}{144}$ | 45 | 469872 | 9215455916 | 222628516313454 | 5992746995783064438 | 172421735348939185816992 |
| $-6^6t_6$ | $0$ | -1 | 1944 | 10066356 | 139857401664 | 2615615263199250 | 57453864811412558112 |
| $-\frac{1}{2}t_7$ | $-\frac{1}{72}$ | 7 | 32859 | 414746092 | 7395891627375 | 157811370338782458 | 3761184845284146266940 |
| $\frac{18}{7}t_8$ | $-\frac{1}{3024}$ | 7 | 54855 | 1034706148 | 24546181658391 | 653902684588247058 | 18687787944102314534628 |

Table 2: Coefficients of $q^k$, $0 \le k \le 6$, in the q-expansion of a solution of $\mathsf{D} = \mathsf{D}_4$.

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