Stabilization for a Nonlinear Beam Equation with Localized Damping *

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Abstract

In this paper, we investigate the exponential decay of the energy of the equation

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u + a(x)\, u_t = 0$$

with boundary clamped condition and a local linear dissipation of the type $a(x)\, u_t$. The method of proof is direct and is based on the multipliers technique and some integral inequalities due to Haraux and we obtain explicit decay rate.

1 Introduction

In this paper we establish the exponential decay of the energy of solutions for the localized damped nonlinear equation

$$u_{tt} + \Delta^2 u - M\left(\int_\Omega |\nabla u|^2\, dx\right)\Delta u + a(x)\, u_t = 0, \tag{1.1}$$

where $M(s)$, $s > 0$, is a nonnegative real function, Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We fix $x_0 \in \mathbb{R}^n$ and we set

$$m(x) = x - x_0, \quad \Gamma_0 = \{x \in \partial\Omega;\, m(x) \cdot \nu(x) > 0\}$$

and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$, where $\nu = \nu(x)$ is the outward normal to $\partial\Omega$.

Let $R(x_0) = \max\limits_{x\in\overline{\Omega}} |m(x)|$, $a \in L^\infty(\Omega)$ and $\nabla a \in (L^\infty(\Omega))^n$ be a nonnegative function at $x \in \Omega$ such that

$$a(x) \geq a_0 > 0 \quad \text{a. e. in } \omega \tag{1.2}$$

where ω is a neighborhood of Γ_0 and a_0 is a positive constant.

When $n = 1$, $M(s) = m_0 + ms$, $m > 0$, equation (1.1) represent the model originally proposed by Woinowsky-Kriger [20], for the transversal vibrations of an extensible beam subject to an axial internal force and $u(x,t)$ is the transverse deflection. If $n = 2$ the equation (1.1) represent the "Berger approximation" of the Von Kárman equations, modelling the nonlinear vibrations of a plate [15], pg. $501 - 507$.

We study (1.1) submitted to boundary clamped conditions described by

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \times \mathbb{R}^+ \tag{1.3}$$

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and initial data

$$u(x,0) = u_0(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega. \tag{1.4}$$

Decay rates of energy of the wave equations with localized damping was studied by Zuazua [19], Nakao [14], Tébou [17] and Martinez [12] by differents methods. The same problem in the context of a Bernoulli-Euler equation (1.1) was investigated by Tucsnak [16], with a linear damping and by Charão [5] considering a nonlinear damping. Both authors obtained estimates of decay considering the local damping effetive in a neighborhood of the whole boundary $\partial\Omega$ and $M(s) = \alpha s$, $\alpha > 0$. In the proof they used a unique continuation result of Kim [9] and a compactness argument, a technique developed by Zuazua [19]. This technique introduce in the estimates constants that are not controllable.

We solve the problem through a method introduced by Tébou [17] for the study of the wave equation which is based on the multipliers technique and on some integral inequalities due to Haraux [7], [8]. As in the proof does not use "compactness-uniqueness" argument the constants that appear in the decay rate are explicit and do not depend on the initial data. Furthermore the damping is effective only in a neighborhood of Γ_0.

The existence and uniqueness of the solutions for (1.1) or similar models has been studied by Ball [1] and Medeiros [13] among others authors. Decay estimates of solutions when the damping term is effective everywhere in Ω has been studied by Ball [2], Biler [3], Brito [4], Pereira [6] and Vasconcelos [18].

2 Notations, Assumptions and Main Results

Throughout this paper, we denote by $|\cdot|$ the norm in $L^2(\Omega)$ and $R(x_0) = \max_{x\in\overline{\Omega}} |m(x)|$.

The existence and uniqueness of solutions for (1.1) is guaranteed by the following result.

Theorem 2.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with regular boundary and $a \in L^\infty(\Omega)$. Then we have:*

1. If $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$, then the problem (1.1), (1.3) and (1.4) admits an unique solution u in the class

$$u \in C\left([0,T]; H^4(\Omega) \cap H_0^2(\Omega)\right) \cap C^1\left([0,T]; H_0^2(\Omega)\right).$$

2. If $u_0 \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, then the problem (1.1) admits an unique solution having the regularity

$$u \in C\left([0,T]; H_0^2(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right).$$

Proof. The result follows by means of the standard semigroup techniques or Faedo-Galerkin procedure. For details see [13] and [18].

The energy associated the (1.1) is given by

$$E(t) = \frac{1}{2}\left|u'(t)\right|^2 + \frac{1}{2}|\Delta u(t)|^2 + \frac{1}{2}\widehat{M}\left(|\nabla u(t)|^2\right),$$

where

$$\widehat{M}(\lambda) = \int_0^\lambda M(\xi)\, d\xi. \tag{2.5}$$

It is easy to verify that the energy E satisfies

$$\frac{d}{dt}E(t) = -\int_\Omega a(x)\left|u'(x,t)\right|^2 dx, \tag{2.6}$$

admitting that $a(x) \geq 0$, a.e. in Ω, we have that the energy is decreasing.

The objective of this work is to study the asymptotic behavior in time of the energy for the solutions of (1.1) when $a(x)\, u'$ is effective only in the subset $\omega \subset \Omega$.

From now on we assume that $M \in C^1[0, +\infty)$ is a increasing function such that

$$M(\lambda) \geq m_0 > 0 \text{ for all } \lambda \in (0, \infty). \tag{2.7}$$

The main result of this paper is the following:

Theorem 2.2. *Consider* $\{u_0, u_1\} \in H_0^2(\Omega) \times L^2(\Omega)$. *Let* ω *be a neighborhood of* Γ_0, $a \in L^\infty(\Omega)$ *satisfying* (1.2) *and* M *satisfying* (2.7). *Then there exists a positive constant* τ_0, *such that*

$$E(t) \leq \left[\exp\left(1 - \frac{t}{\tau_0}\right)\right] E(0),\ \forall\, t \geq 0, \tag{2.8}$$

where τ_0 *is independent of the initial data.*

The following Lemma reduces the proof of Theorem 2.2 to the proof of an appropriate estimate.

Lemma 2.1. *Let* $E : [0,\infty[\longrightarrow [0,\infty[$ *be a nonincreasing locally absolutely continuos function and assume that there exists a constant* $C > 0$ *such that*

$$\int_s^\infty E(t)\, dt \leq CE(s) \quad \forall s \in [0, \infty[. \tag{2.9}$$

Then

$$E(t) \leq \left[\exp\left(1 - \frac{t}{C}\right)\right] E(0),\ \forall\, t \geq C. \tag{2.10}$$

Proof. The proof is due to Haraux and can be found in [7], [8] and Komornik [10].

Let us consider s and T real numbers such that $0 \leq s \leq T < \infty$ and $Q_s = \Omega \times]s, T[$ the cylindrical domain of $\mathbb{R}^{n+1}$ with lateral bounded $\Sigma_s = \Gamma \times]s, T[$.

The estimate (2.9) is obtained by means of the following lemma:

Lemma 2.2. *Let u be the solution of* (1.1), $q \in \left(W^{2,\infty}(\Omega)\right)^n$, $\xi \in W^{2,\infty}(\Omega)$ *and* $\alpha \in \mathbb{R}$. *Then we have the identities,*

$$\begin{aligned}
&\frac{1}{2}\int_{Q_s} (div\ q - \alpha)\left(|u'|^2 - |\Delta u|^2\right) dxdt + \left(u'(t), q \cdot \nabla u(t) + \frac{1}{2}\alpha u(t)\right)\Bigg|_s^T + \\
&+2\int_{Q_s} \Delta u \frac{\partial q_k}{\partial x_j}\frac{\partial^2 u}{\partial x_k \partial x_j} dxdt + \int_{Q_s} a(x)\, u' \left(q \cdot \nabla u + \frac{1}{2}\alpha u\right) dxdt + \\
&-\frac{1}{2}\int_{Q_s} \{div\ q - \alpha\}\, M\left(|\nabla u|^2\right)|\nabla u|^2\, dxdt + \int_{Q_s} M\left(|\nabla u|^2\right)\nabla u \cdot \nabla q_k \frac{\partial u}{\partial x_k} dxdt + \\
&+\int_{Q_s} \Delta u\, \Delta q \cdot \nabla u\, dxdt = \frac{1}{2}\int_{\Sigma_s} q \cdot \nu\, |\Delta u|^2\, d\Sigma,
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\left(u'(t), \xi u(t)\right)\Big|_s^T - \int_{Q_s} \xi\left(|u'|^2 - |\Delta u|^2\right) dxdt + \int_{Q_s} \Delta u\, \Delta\xi\, u\, dxdt + \\
&+2\int_{Q_s} \Delta u\, (\nabla u \cdot \nabla \xi)\, dxdt + \int_{Q_s} M\left(|\nabla u|^2\right)\nabla u \cdot \nabla \xi\, u\, dxdt + \\
&+\int_{Q_s} M\left(|\nabla u|^2\right)|\nabla u|^2\, \xi dxdt + \int_{Q_s} a(x)\, u' \xi u dxdt = 0.
\end{aligned} \tag{2.12}$$

with the summation convention of repeated indices.

Proof. The first identity is obtained multiplying the equation in (1.1) by $q \cdot \nabla u + \frac{1}{2}\alpha u$ and integrating on Q_s. The identity in (2.12) is obtained by means of the multiplier $u\xi$.

3 Proof of Theorem 2.2

The proof consists in proving the estimate (2.10) and will be done in several steps. It is enought to consider $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$, $u_1 \in H_0^2(\Omega)$ and then to use density.

Step 1. Applying (2.11) with $\alpha = n - 1$, $q(x) = m(x)$ and observing that $div\ m = n$ and $\Delta q = 0$, we find

$$\begin{aligned}
&\frac{1}{2}\int_s^T \left(|u'(t)|^2 + |\Delta u(t)|^2 + M\left(|\nabla u(t)|^2\right)|\nabla u(t)|^2\right) dt + \int_s^T |\Delta u(t)|^2 = \\
&= \frac{1}{2}\int_{\Sigma_s} m(x) \cdot \nu\, |\Delta u|^2\, d\Sigma - \left(u', m(x) \cdot \nabla u + \frac{(n-1)}{2}u\right)\Bigg|_s^T - \\
&-\int_{Q_s} a(x)\, u' \left(m(x) \cdot \nabla u + \frac{(n-1)}{2}u\right) dxdt.
\end{aligned} \tag{3.13}$$

Since M is increasing we have that $\widehat{M}(s) \leq sM(s)$, where $\widehat{M}(s)$ was defined in (2.5), and therefore from (3.13) follows that

$$\begin{aligned}
&\int_s^T E(t)\, dt \leq -\int_{Q_s} a(x)\, u' \left(m(x) \cdot \nabla u + \frac{(n-1)}{2}u\right) dxdt - \\
&- \left(u'(t), m(x) \cdot \nabla u(t) + \frac{(n-1)}{2}u(t)\right)\Bigg|_s^T + \frac{1}{2}\int_{\Sigma_s} m(x) \cdot \nu\, |\Delta u|^2\, d\Sigma.
\end{aligned} \tag{3.14}$$

Setting,

$$X(t)=\left(u'(t),m(x)\cdot\nabla u(t)+\frac{(n-1)}{2}u(t)\right),$$

follows that

$$|X(t)|\le\frac{R(x_0)}{2}\int_\Omega\left|u'\right|^2dx+\frac{R(x_0)}{2}K_0^2\int_\Omega|\Delta u|^2dx\le R(x_0)\max\left\{1,K_0^2\right\}E(t),$$

where K_0 is the immersion's constant of the $H_0^2(\Omega)\hookrightarrow H_0^1(\Omega)$.

On the other hand, from (2.6) and Holder´s Inequality

$$-\int_{Q_s}a(x)u'\left(m(x)\cdot\nabla u+\frac{(n-1)}{2}u\right)dxdt\le$$

$$\le\|a\|_{L^\infty(\Omega)}R^2(x_0)K_0^2E(s)+\frac{1}{2}\int_s^TE(t)\,dt.$$

Defining $\Sigma_s(x_0)=\Gamma_0\times(s,T)$ and remembering that $m(x)\cdot\nu(x)>0$ on Γ_0 we obtain

$$\frac{1}{2}\int_{\Sigma_s}m(x)\cdot\nu(x)|\Delta u|^2d\Sigma\le\frac{1}{2}\int_{\Sigma_s(x_0)}m(x)\cdot\nu(x)|\Delta u|^2d\Sigma\le\frac{R(x_0)}{2}\int_{\Sigma_s(x_0)}|\Delta u|^2d\Sigma.$$

Substituting the above estimates in (3.14) we find

$$\frac{1}{2}\int_s^TE(t)\,dt\le C_1E(s)+\frac{R(x_0)}{2}\int_{\Sigma_s(x_0)}|\Delta u|^2d\Sigma,\tag{3.15}$$

where $C_1=R(x_0)\left(2\max\left\{1,K_0^2\right\}+\|a\|_{L^\infty(\Omega)}R(x_0)K_0\right)$.

Step 2. Estimate of $\int_{\Sigma_s(x_0)}|\Delta u|^2d\Sigma$.

Consider $h\in\left(W^{2,\infty}(\Omega)\right)^n$ such that

$$h=\nu\text{ on }\Gamma_0,\ h\cdot\nu\ge0\text{ on }\Gamma,\ h=0\text{ in }\Omega\setminus\widehat{\omega},$$

where $\widehat{\omega}$ is another neighborhood of Γ_0 strictly contained in ω. For the construction of such function to see Lions [11].

Choosing $\alpha=0$ and $q=h$ in the identity in (2.11) we obtain

$$\begin{aligned}&\int_{\Sigma_s}h(x)\cdot\nu(x)|\Delta u|^2d\Sigma=\int_{\widehat{\omega}\times(s,T)}div\,h\left(|u'|^2-|\Delta u|^2\right)dxdt+2\int_{\widehat{\omega}}u'(t)\,h\cdot\nabla u(t)\,dx|_s^T+\\&+2\int_{\widehat{\omega}\times(s,T)}\Delta u\,\Delta h\cdot\nabla u\,dxdt+2\int_{\widehat{\omega}\times(s,T)}a(x)u'(h\cdot\nabla u)\,dxdt-\\&-\int_{\widehat{\omega}\times(s,T)}div\,h\,M\left(|\nabla u|^2\right)|\nabla u|^2dxdt+2\int_{\widehat{\omega}\times(s,T)}M\left(|\nabla u|^2\right)\nabla u\cdot\nabla h_k\frac{\partial u}{\partial x_k}dxdt+\\&+4\int_{\widehat{\omega}\times(s,T)}\Delta u\,\frac{\partial h_k}{\partial x_j}\frac{\partial^2u}{\partial x_k\partial x_j}\,dxdt\le\|div\,h\|_{L^\infty(\Omega)}\int_{\widehat{\omega}\times(s,T)}\left(|u'|^2+|\Delta u|^2+M\left(|\nabla u|^2\right)|\nabla u|^2\right)dxdt+\\&+2\int_{\widehat{\omega}}u'(t)\,h\cdot\nabla u(t)\,dx|_s^T+2\int_{\widehat{\omega}\times(s,T)}\Delta u\,\Delta h\cdot\nabla u\,dxdt+2\int_{\widehat{\omega}\times(s,T)}a(x)u'(h\cdot\nabla u)\,dxdt+\\&2\int_{\widehat{\omega}\times(s,T)}M\left(|\nabla u|^2\right)\nabla u\cdot\nabla h_k\frac{\partial u}{\partial x_k}dxdt+4\int_{\widehat{\omega}\times(s,T)}\Delta u\,\frac{\partial h_k}{\partial x_j}\frac{\partial^2u}{\partial x_k\partial x_j}\,dxdt\end{aligned}\tag{3.16}$$

Observing that

- $2\int_{\widehat{\omega}\times(s,T)} \Delta u\, \Delta h \cdot \nabla u\, dxdt \leq 2\,\|\Delta h\|_{L^\infty(\Omega)}\, K_0 \int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\, dxdt.$

- $4\int_{\widehat{\omega}\times(s,T)} \Delta u\, \frac{\partial h_k}{\partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_j}\, dxdt \leq 2\left\|\frac{\partial h_k}{\partial x_j}\right\|_{L^\infty(\Omega)} (1+K_1) \int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\, dxdt.$

- $\left|2\int_{\widehat{\omega}} u'(t)\, h\cdot \nabla u(t)\, dx\right| \leq 2\,\|h\|_{L^\infty(\Omega)} \left(\int_{\widehat{\omega}} \left|u'(t)\right|^2 dx\right)^{1/2} \left(\int_{\widehat{\omega}} |\nabla u(t)|^2\, dx\right)^{1/2} \leq 4K_0\,\|h\|_{L^\infty(\Omega)}\, E(t)$. Therefore
$$2\int_{\widehat{\omega}} u'(t)\, h\cdot \nabla u(t)\, dx\Big|_s^T \leq 8K_0\,\|h\|_{L^\infty(\Omega)}\, E(s)\,.$$

- $2\int_{\widehat{\omega}\times(s,T)} a(x)\, u'\,(h\cdot\nabla u)\, dxdt \leq \int_s^T 2\sqrt{R(x_0)}\,\|a\|^{1/2}_{L^\infty(\Omega)} \left|E'(t)\right|^{1/2} \|h\|_{(L^\infty(\Omega))^n}\, K_0\, \sqrt{2}\left(\frac{E(t)}{R(x_0)}\right)^{1/2} dt \leq$

$\leq 4R(x_0)\, K_0^2\, \|a\|_{L^\infty(\Omega)}\, \|h\|^2_{(L^\infty(\Omega))^n}\, E(s) + \frac{1}{2R(x_0)}\int_s^T E(t)\, dt.$
Taking in consideration that
$$\sum_{k=1}^n \left|\frac{\partial u}{\partial x_k}\right| \leq n\,|\nabla u|$$
and $a(x) \geq a_0 > 0$ in $\widehat{\omega}$, we have:
$$2\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right) \nabla u\cdot \nabla h_k \frac{\partial u}{\partial x_k} dxdt \leq 2\,\|\nabla h_k\|_{L^\infty(\Omega)}\, n \int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right) |\nabla u|^2\, dxdt$$
and
$$\int_{\widehat{\omega}\times(s,T)} \left|u'\right|^2 dxdt \leq \frac{1}{a_0}\int_{\widehat{\omega}\times(s,T)} a(x)\left|u'\right|^2 dxdt \leq \frac{1}{a_0}\int_s^T \left|E'(t)\right| dt \leq \frac{1}{a_0} E(s)\,.$$

By the construction of h follows
$$\int_{\Sigma_s(x_0)} |\Delta u|^2\, d\Sigma = \int_{\Sigma_s(x_0)} (h(x)\cdot\nu(x))\, |\Delta u|^2\, d\Sigma \leq \int_{\Sigma_s} (h(x)\cdot\nu(x))\, |\Delta u|^2\, d\Sigma. \tag{3.17}$$

Multipling (3.16) by $\frac{R(x_0)}{2}$, as a consequence of (3.17) and the estimates above we conclude
$$\begin{aligned} \frac{R(x_0)}{2}\int_{\Sigma_s(x_0)} |\Delta u|^2\, d\Sigma &\leq C_2 E(s) + C_3 \int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\, dxdt + \\ &+ C_4 \int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right) |\nabla u|^2\, dxdt + \frac{1}{4}\int_s^T E(t)\, dt, \end{aligned} \tag{3.18}$$

where C_2, C_3 and C_4 are constants independents of the initial data.

Substituting (3.18) in (3.15) we find

$$\begin{aligned}\frac{1}{2}\int_s^T E(t)\,dt &\le (C_1+C_2)\,E(s) + C_3\int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt + \\ &+C_4\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt \le (C_1+C_2)\,E(s) + \\ &+C_5\left(\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt + \int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt\right)\end{aligned} \tag{3.19}$$

where $C_5 = \max\{C_3, C_4\}$.

Step 3. Estimate of $\int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt$ and $\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt$.

We introduce the function η, that satisfies

$$\eta \in W^{2,\infty}(\Omega),\ 0 \le \eta \le 1,\ \eta = 1 \text{ in } \widehat{\omega},\ \eta = 0 \text{ in } \Omega \setminus \omega.$$

Applying (2.12) with $\xi = \eta^2$ we have

$$\begin{aligned}&\left(u'(t), \eta^2 u(t)\right)\Big|_0^T - \int_{Q_s} \eta^2\left(|u'|^2 - |\Delta u|^2\right)dxdt + \int_{Q_s} \Delta u\,\Delta\eta^2\,u\,dxdt \\ &+2\int_{Q_s} \Delta u\left(\nabla u\cdot\nabla\eta^2\right)dxdt + \int_{Q_s} M\left(|\nabla u|^2\right)|\nabla u|^2\,\eta^2 dxdt + \\ &+\int_{Q_s} M\left(|\nabla u|^2\right)\nabla u\cdot\nabla\eta^2\,u\,dxdt + \int_{Q_s} a(x)\,u'\eta^2 u dxdt = 0.\end{aligned} \tag{3.20}$$

From (3.20) and properties of η follows that

$$\begin{aligned}&\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt + \int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt = \\ &-\left(u'(t), \eta^2 u(t)\right)\Big|_s^T + \int_{Q_s} \eta^2|u'|^2\,dxdt - \\ &-2\int_{Q_s} \Delta u\left(\nabla u\cdot\nabla\eta^2\right)dxdt - \int_{Q_s} \Delta u\,\Delta\eta^2\,u\,dxdt - \\ &\int_{Q_s} M\left(|\nabla u|^2\right)\nabla u\cdot\nabla\eta^2\,u\,dxdt - \int_{Q_s} a(x)\,u'\eta^2 u dxdt.\end{aligned} \tag{3.21}$$

Now, observe that,

- $2\int_{Q_s} \Delta u\left(\nabla u\cdot\nabla\eta^2\right)dxdt \le \frac{1}{4}\int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt + 4\left\|\nabla\eta^2\right\|_\infty^2 \int_{\widehat{\omega}\times(s,T)} |\nabla u|^2\,dxdt$

- $\int_{Q_s} \Delta u\,\Delta\eta^2\,u\,dxdt \le \left\|\Delta\eta^2\right\|_\infty^2 \int_{\widehat{\omega}\times(s,T)} |u|^2\,dxdt + \frac{1}{4}\int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt \le$

$\le \frac{1}{m_0}\left\|\Delta\eta^2\right\|_\infty^2 \int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|u|^2\,dxdt + \frac{1}{4}\int_{\widehat{\omega}\times(s,T)} |\Delta u|^2\,dxdt$

- $\int_{Q_s} M\left(|\nabla u|^2\right)\nabla u\cdot\nabla\eta^2\, u\,dxdt \le \frac{1}{2}\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt+$

$+\frac{\left\|\nabla\eta^2\right\|_\infty^2}{2}\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|u|^2\,dxdt.$

- $\int_{Q_s} a(x)\,u'\eta^2 u dxdt \le \int_s^T \sqrt{\|a\|_\infty}\left|E'(t)\right|^{1/2} K_1\,|\Delta u(t)|\,dt \le$

$\le 4\|a\|_\infty K_1^2 C_5 E(s) + \frac{1}{8C_5}\int_s^T E(t)\,dt$, where K_1 is the immersion's constant of the $H_0^2(\Omega)\hookrightarrow L^2(\Omega)$.

- $\int_{Q_s}\eta^2|u'|^2\,dxdt \le \frac{1}{a_0}\int_{Q_s}\eta^2 a(x)|u'|^2\,dxdt \le \frac{1}{a_0}E(s).$

- $\left(u'(t),\eta^2 u(t)\right)\Big|_s^T \le 2K_2E(s)$, where $K_2=\max\left\{1,K_1^2\right\}$.

Using the estimates above in (3.21) we obtain

$$\begin{aligned}&\frac{C_5}{2}\left(\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|\nabla u|^2\,dxdt+\int_{\widehat{\omega}\times(s,T)}|\Delta u|^2\,dxdt\right)\le\\ &\le\left(4K_1^2C_5\|a\|_\infty+\frac{C_5}{a_0}+2C_5K_2\right)E(s)+\\ &+4C_5\left\|\nabla\eta^2\right\|_\infty^2\int_{\widehat{\omega}\times(s,T)}|\nabla u|^2\,dxdt+\left(\frac{C_5}{m_0}\left\|\Delta\eta^2\right\|_\infty^2+\frac{C_5}{2}\left\|\nabla\eta^2\right\|_\infty^2\right)\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|u|^2\,dxdt+\\ &+\frac{1}{8}\int_s^T E(t)\,dt.\end{aligned}\tag{3.22}$$

Substituting (3.22) in (3.19) we find

$$\begin{aligned}\int_s^T E(t)\,dt &\le C_6E(s)+C_7\int_{\widehat{\omega}\times(s,T)} M\left(|\nabla u|^2\right)|u|^2\,dxdt+C_8\int_{\widehat{\omega}\times(s,T)}|\nabla u|^2\,dxdt\le\\ &\le C_6E(s)+C_7\int_{\widehat{\omega}\times(s,T)} a(x)\,M\left(|\nabla u|^2\right)|u|^2\,dxdt+C_8\int_{\widehat{\omega}\times(s,T)} a(x)\,|\nabla u|^2\,dxdt,\end{aligned}\tag{3.23}$$

because $a(x)\ge a_0>0$ in $\omega\supset\widehat{\omega}$.

Let us consider a special multiplier to absorb the terms of the right side of (3.23). We introduce $z(t)\in H_0^1(\Omega)\cap H^2(\Omega)$ solution of the problems

$$\left|\begin{array}{l}-\Delta z=a(x)\,u \text{ in } \Omega\\ z=0 \text{ on } \Gamma,\end{array}\right.\tag{3.24}$$

and

$$\left|\begin{array}{c}-\Delta z'=a(x)\,u' \text{ in } \Omega\\ z'=0 \text{ on } \Gamma.\end{array}\right.\tag{3.25}$$

Elementary calculations show that

$$\begin{aligned} \int_\Omega |\nabla z|^2\,dx &\le C\int_\Omega |u|^2\,dx \\ \int_\Omega |\nabla z'|^2\,dx &\le C\int_\Omega a(x)\,|u'|^2\,dx. \end{aligned} \tag{3.26}$$

Multiplying the equation in (1.1) by z and integrating by parts on Q_s we find

$$\int_{Q_s} u'' z dxdt + \int_{Q_s} \Delta u \Delta z dxdt - \int_{Q_s} M\left(|\nabla u|^2\right) u\Delta z dxdt + \int_{Q_s} a(x)\,u' z dxdt = 0. \tag{3.27}$$

Now from (3.24) we obtain

$$\begin{aligned} &-\int_{Q_s} M\left(|\nabla u|^2\right)\Delta z u dxdt = \int_{Q_s} a(x)\,M\left(|\nabla u|^2\right)|u|^2\,dxdt, \\ &\int_{Q_s} \Delta z \Delta u dxdt = -\int_{Q_s} \Delta u\; a(x)\,u dxdt = \int_{Q_s} \nabla u \nabla\left(a(x)\,u\right) dxdt \le \\ &\le C\int_{Q_s} a(x)\,|\nabla u|^2\,dxdt. \end{aligned} \tag{3.28}$$

Substituting (3.28) in (3.27) we conclude that

$$\int_{Q_s} a(x)\,M\left(|\nabla u|^2\right)|u|^2\,dxdt + \int_{Q_s} a(x)\,|\nabla u|^2\,dxdt \le -\int_\Omega \; u' z dx\Big|_s^T + \int_{Q_s} u' z' dxdt - \int_{Q_s} a(x)\,u' z dxdt. \tag{3.29}$$

From Poincaré´s Inequality and (3.26) we get

$$\begin{aligned} \left|-\int_\Omega u' z dx\right| &\le \frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{K_0}{2}\int_\Omega |\nabla z|^2\,dx \le \\ &\le \frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{K_0 C}{2}\int_\Omega |u|^2\,dx \le \\ &\le \frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{K_0 K_1 C}{2}\int_\Omega |\Delta u|^2\,dx \le C_9 E(t). \end{aligned}$$

Therefore,

$$-\int_\Omega u' z dx\Big|_s^T \le 2C_9 E(s).$$

Using (3.24), (3.26) and the immersion of $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$ we have

$$\begin{aligned} \int_{Q_s} a(x)\,u' z dxdt &= \int_s^T \left\{\left(\int_\Omega a(x)^2\,|u'|^2\,dx\right)^{1/2}\left(\int_\Omega |z|^2\,dx\right)^{1/2}\right\} dt \le \\ &\le C\,\|a\|_\infty^{1/2}\int_s^T \frac{1}{\varepsilon}\,|E'(t)|^{1/2}\cdot \varepsilon E(t)^{1/2}\,dt \le \\ &\le \frac{C\,\|a\|_\infty}{\varepsilon}\int_s^T |E'(t)|\,dt + \frac{\varepsilon}{2}\int_s^T E(t)\,dt. \end{aligned}$$

From (3.26) follows that

$$\int_{Q_s} u'z'dxdt \leq \frac{\varepsilon}{4}\int_s^T \int_\Omega |u'|^2 \, dxdt + \frac{1}{\varepsilon}\int_s^T \int_\Omega |z'|^2 \, dxdt \leq$$
$$\leq \frac{\varepsilon}{2}\int_s^T E(t)\,dt + \frac{CK_1}{\varepsilon}\int_s^T \int_\Omega a(x)\,|u'|^2 \, dxdt \leq$$
$$\leq \frac{\varepsilon}{2}\int_s^T E(t)\,dt + \frac{CK_1}{\varepsilon}\int_s^T |E'(t)|\,dt \leq \frac{\varepsilon}{2}\int_s^T E(t)\,dt + C_{10}E(s)\,.$$

Substituting the estimates above in (3.29) and backing in (3.23) follows that

$$(1-\varepsilon)\int_s^T E(t)\,dt \leq C_{11}E(s)\,. \tag{3.30}$$

Choosing $0 < \varepsilon < 1$ and taking the limit when $T \to +\infty$ we obtain the estimate (2.9) and this concludes the proof.

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