Symmetric spaces and partial hyperbolicity *

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Resumo

Falaremos a respeito da hiperbolicidade e hiperbolicidade parcial do fluxo geodésico de espaços simétricos de tipo não-compacto.

Abstract

We will talk about hyperbolicity and partial hyperbolicity for the geodesic flow of locally symmetric spaces of noncompact type

1 Introduction

In the sixties, Anosov showed that the geodesic flow of a Riemannian manifold of negative curvature is hyperbolic. One classical example of hyperbolic system is the geodesic flow of a Riemannian manifold of constant curvature -1. There are other nice examples: locally symmetric manifolds with negative curvature. We are going to show they are hyperbolic and partially hyperbolic with the help of a tecnique which turns the proof of hyperbolicity and partial hyperbolicity very elegant.

In section 2 we introduce definitions, the criteria for hiperbolicity we are going to use and a proof that the geodesic flow of a Riemannian manifold with negative curvature is hyperbolic.

In section 3 we introduce definitions and properties of symmetric spaces and locally symmetric manifolds.

In section 4 we show that the geodesic flow of a locally symmetric manifold of nonconstant negative curvature is partially hyperbolic.

2 Cone field criteria

Before given the results, we need to introduce a few definitions.

A partially hyperbolic flow $\phi_t : M \to M$ in the manifold M generated by the vector field $X : M \to TM$ is a flow such that its quotient bundle $TM/\langle X\rangle$ (assuming that X has not singularities) have an invariant splitting $TM/\langle X\rangle = E^s \oplus E^c \oplus E^u$ such that these subbundles are non trivial and with the following properties:

$$\begin{aligned} d\phi_t(x)(E^s(x)) &= E^s(\phi_t(x)),\\ d\phi_t(x)(E^c(x)) &= E^c(\phi_t(x)),\\ d\phi_t(x)(E^u(x)) &= E^u(\phi_t(x)), \end{aligned} \tag{2.1}$$

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$$||d\phi_t(x)|_{E^s}|| \leq C\exp(t\lambda),$$
$$||d\phi_{-t}(x)|_{E^u}|| \leq C\exp(t\lambda),$$
$$C\exp(t\mu) \leq ||d\phi_t(x)|_{E^c}|| \leq C\exp(-t\mu),$$

for $\lambda < \mu < 0 < C$.

Let (M,g) be a Riemannian manifold, $\pi : TM \to M$ its tangent bundle, $\phi_t : TM \to TM$ be its geodesic flow. For any $v \in TM$, $\phi_t(v) = \gamma_v(t)$, where γ_v is the geodesic such that $\gamma_v(0) = \pi(v)$ and $\gamma_v'(0) = v$. Any geodesic has constant velocity and $\gamma_{\frac{v}{\lambda}}(t) = \gamma_v(\lambda t)$, for any positive real number λ, so we consider only the geodesics with velocity one, i.e., $UM = \{v \in TM : g(v,v) = 1\}$.

The double tangent bundle TTM is isomorphic to the vector bundle

$$\mathcal{E} \to TM,$$

$\mathcal{E} = \pi^*TM \oplus \pi^*TM$, with fiber $\mathcal{E}_v = T_{\pi(v)}M \oplus T_{\pi(v)}M$. We define the isomorphism as

$$\mathcal{I} : TTM \to \mathcal{E} : Z \to ((\pi \circ V)'(0), \frac{DV}{dt}(0)),$$

where V is a curve on TM such that $V'(0) = Z$. The Sasaki metric in the double tangent bundle is the pull-back of the metric $\widetilde{g}$ on $\mathcal{E}$:

$$\widetilde{g}_v((\eta_1,\varsigma_1),(\eta_2,\varsigma_2)) = g_{\pi(v)}(\eta_1,\eta_2) + g_{\pi(v)}(\varsigma_1,\varsigma_2)$$

where $(\eta_1,\varsigma_1),(\eta_2,\varsigma_2) \in T_{\pi(v)}M \oplus T_{\pi(v)}M$.

There is a subbundle $E(UM) \subset T(UM)$ transversal to the geodesic flow which is identified with the vector bundle $\mathcal{E}' \to UM$, $\mathcal{E}'_v \subset \mathcal{E}_v$, for all $v \in UM$, whose fiber at v is $v^\perp \oplus v^\perp$, where $v^\perp = \{w \in T_{\pi(v)}M : g(w,v) = 0\}$. The derivative of the geodesic flow, given the identification of TTM and $\mathcal{E}$ is

$$d_v\phi_t(\eta,\varsigma) = (J(t), J'(t)),$$

where $J(t) \in T_{\phi_t(v)}M$ is a Jacobi field [Ba1],[P], i.e.,

$$J''(t) + R(\phi_t(v), J(t))\phi_t(v) = 0.$$

Let $p : E(UM) \to UM$ be the subbundle transversal to the geodesic flow of (M,g). Let $Q : E(UM) \to \mathbb{R}$ be a nondegenerate quadractic form of constant signature (l,m). Let $\mathcal{C}_+(x,v) = \{\eta \in \xi(x,v) : Q_{(x,v)}(\eta) > 0\}$ be its positive cone, $\mathcal{C}_-(x,v) = \{\eta \in \xi(x,v) : Q_{(x,v)}(\eta) < 0\}$ be its negative cone, $\mathcal{C}_0(x,v) = \{\eta \in \xi(x,v) : Q_{(x,v)}(\eta) = 0\}$ be their boundary and $\overline{\mathcal{C}}_+ = \mathcal{C}_+ \cup \mathcal{C}_0$, $\overline{\mathcal{C}}_- = \mathcal{C}_- \cup \mathcal{C}_0$. The criteria says the following:

Lemma 2.1. *If*

$$\frac{d}{dt}\mathcal{Q}(\eta,\varsigma) > 0$$

for all $(\eta,\varsigma) \in \overline{\mathcal{C}}_+(x,v)$, $(x,v) \in UM$, *the the geodesic flow has a partially hyperbolic splitting* $E(UM) = E^s \oplus E^c \oplus E^u$, $dim(E^\sigma) = l$, $\sigma = s,u$.

Demonstração. See the proof in [W].

□

2.1 Negatively curved manifolds

In the negatively curved case, the lemma imples it is hyperbolic, i.e., there is a partially hyperbolic splitting with trivial central subbundle: $E^c = \{0\}$. Suppose $K \leq -\alpha^2$, for a positive real number α. Define the quadratic form as

$$Q(\eta, \varsigma) = g(\eta, \varsigma).$$

It is easy to show that it is a quadratic form with signature $(n-1, n-1)$.

Theorem 2.1. *Let (M, g) be a Riemannian manifold with negative sectional curvature. Then its geodesic flow is hyperbolic.*

Demonstração.

$$\frac{d}{dt} g(\eta, \varsigma) = g(\varsigma, \varsigma) - R(v, \eta, v, \eta) \geq g(\varsigma, \varsigma) + \alpha^2 g(\eta, \eta) > 0.$$

for any $(\eta, \varsigma) \in \overline{\mathcal{C}}_+(x, v)$, $(x, v) \in UM$. □

3 Symmetric spaces of nonpositive curvature

In this section we give a brief introdution of the subject of symmetric and locally symmetric spaces [E2],[E3],[J], and prove that the geodesic flow of a compact locally symmetric spaces of nonpositive curvature is partially hyperbolic only if it has nonconstant negative curvature.

Definition 3.1. *A Riemmanian manifold (M, g)*

a. M is called symmetric if for all $x \in M$ there is an isometry $\sigma_x : M \to M$ such that

$$\sigma_x(x) = x, d\sigma_x(x) = -id_{T_x M};$$

b. M is called locally symmetric if for all $x \in M$ there is a radius r, a ball $B_r(x)$ centered at x and an isometry $\sigma_x : B_r(x) \to B_r(x)$ such that

$$\sigma_x(x) = x, d\sigma_x(x) = -id_{T_x M}.$$

Proposition 3.1. *Let (M, g) be a symmetric space.*

- *If γ is a geodesic with $\gamma(0) = x$ then $\sigma_x(\gamma(t)) = \gamma(-t)$;*
- *M is complete;*

Demonstração.

An isometry takes geodesics to geodesics, so $c(t) = \sigma_x(\gamma(t))$ is a geodesic such that $c(0) = \sigma_x(\gamma(0)) = x$ and $c'(0) = d\sigma_x(x)\gamma'(0) = -\gamma'(0)$. By the property of uniqueness of geodesics, $c(t) = \gamma(-t)$.

If M is not complete, let $\gamma : [0, T) \to M$ be the geodesic γ defined for its maximal domain. Applying $\sigma_{\gamma(T-\epsilon)}$ to γ and item a. enables us to extend the domain of definition to $[0, 2T - 2\epsilon)$ for a small enough ϵ such that $2T - 2\epsilon > T$. □

Definition 3.2. *Let (M, g) be a symmetric space, then we define for every geodesic $\gamma : \mathbb{R} \to M$ the map*

$$\tau_t : M \to M,$$

such that

$$\tau_t = \sigma_{\gamma(\frac{t}{2})} \circ \sigma_{\gamma(0)}.$$

Proposition 3.2. *Let (M, g) be a symmetric space and γ a geodesic*

- *τ_t translates γ: $\tau_t(\gamma(s)) = \gamma(t + s)$;*
- *$d\tau_t(\gamma(s))\xi$ is the parallel translation of $\xi \in \tau_{\gamma(s)}$ to $\tau_{\gamma(s+t)}$ along γ;*
- *τ_t is a one-parameter group of isometries: $\tau_{s+t} = \tau_s \circ \tau_t$.*

Definition 3.3. $I_0(M) = \{g_t$ *for* $t \in \mathbb{R} : s \to g_s$ *is a group homomorphism from* $\mathbb{R}$ *to* $G\}$

Corollary 3.1. *Geodesics in the symmetric space (M, g) are images of one-parameter group of isometries.*

Corollary 3.2. *Let $I_0(M)$ be the connected component of the group of isometries of M which contains the identity, then $I_0(M)$ acts transitively on M.*

Remark 3.1. *A Riemmanian manifold with transitive group of isometries is called homogeneous.*

By definition

$$(\nabla_X R)(Y, Z)W = X(R(Y, Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y, Z)\nabla_X W.$$

Then if Y, Z, W are parallel then

$$(\nabla_X R)(Y, Z)W = X(R(Y, Z)W).$$

Proposition 3.3. *Let (M, g) be locally symmetric. Then*

$$\nabla R = 0.$$

Demonstração. Let γ be a geodesic, with $\gamma(0) = x$ and $\gamma(T) = y$. Let τ_T be the transvection along γ. Let X, Y, Z, W be parallel vector fields along γ. Then

$$g(R(Y(q), Z(q))W(q), U(q)) = g(R(d\tau_T Y(q), d\tau_T Z(q))d\tau_T W(q), d\tau_T U(q))$$

$$= g(R(Y(p), Z(p))W(p), U(p)) \Rightarrow X(R(Y, Z)W, U) = 0.$$

□

One can show that if (M, g) is a compact Riemannian manifold, then the following properties are equivalent to being locally symmetric:

- $\nabla R \equiv 0$,
- if $X(t)$, $Y(t)$ and $Z(t)$ are parallel vector fields along $\gamma(t)$, then $R(X(t), Y(t))Z(t)$ is also a parallel vector field along $\gamma(t)$.

Each simply connected symmetric space M is the quotient of the Lie group G of isometries of M with a left invariant metric by its maximal compact subgroup K: $M = G/K$. Each compact locally symmetric space N is the quotient of a simply connected symmetric space M and a cocompact lattice Γ of G acting on M discretely, without fixed points, and isometrically, such that $N = M/\Gamma$ [E2],[E3],[J].

Proposition 3.4. *Let N be a locally symmetric space, $p \in N$, $v \in T_pN$, c geodesic such that $c(0) = p$, $c'(0) = v$, there are $v_1, \dots, v_{n-1}$ an orthogonal basis of eigenvectors of $R_{c'(0)}$ orthogonal to v with eigenvalues $\rho_1, \dots, \rho_{n-1}$, and parallel vector fields $v_1(t), \dots, v_{n-1}(t)$ along c such that $v_i(0) = v_i$. Then the Jacobi fields along c are linear combinations of the following Jacobi fields*

$$c_{\rho_j}(t)v_j(t) \text{ and } s_{\rho_j}(t)v_j(t),$$

where

$$c_\rho(t) := \begin{cases} \cos\sqrt{\rho}t, \rho > 0, \\ \cosh\sqrt{-\rho}t, \rho < 0, \\ 1, \rho = 0, \end{cases} \quad s_\rho(t) := \begin{cases} \dfrac{1}{\sqrt{\rho}} \sin\sqrt{\rho}t, \rho > 0, \\ \dfrac{1}{\sqrt{-\rho}} \sinh\sqrt{-\rho}t, \rho < 0, \\ t, \rho = 0. \end{cases}$$

The proof of the proposition relies on the two facts: $R_v : T_pN \to T_pN : w \to R(v, w)v$ is a self-adjoint map and the curvature tensor is parallel [J].

Definition 3.4. *Let $M = G/K$ be a symmetric space, where G is the Lie group of isometries of M and K the maximal compact subgroup of G. Let $\mathfrak{g}$ be the algebra of Killing fields on the symmetric space M and $p \in M$. Define*

$$\mathfrak{k} := \{X \in \mathfrak{g} : X(p) = 0\},$$

$$\mathfrak{p} := \{X \in \mathfrak{g} : \nabla X(p) = 0\}.$$

For these subspaces of $\mathfrak{g}$, $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ and $\mathfrak{k} \cap \mathfrak{p} = \{0\}$, and T_pM identifies with $\mathfrak{p}$.

Remark 3.2. *In fact the Lie algebra of G is $\mathfrak{g}$ and the Lie algebra of K is $\mathfrak{k}$.*

Definition 3.5. *Given $p \in M$, we define the involution $\phi_p(g) : G \to G : g \to \sigma_p \circ g \circ \sigma_p$. Then, we obtain $\theta_p : d\phi_p : \mathfrak{g} \to \mathfrak{g}$. Since $\theta_p^2 = id$ and θ_p preserves the lie brackets, the properties of this subspaces of $\mathfrak{g}$ are:*

i. $\theta_{p|\mathfrak{k}} = id$,

ii. $\theta_{p|\mathfrak{p}} = -id$,

iii. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$,

Proposition 3.5. *With the identification $T_pM \cong \mathfrak{p}$ the curvature tensor of M satisfies*

$$R(X,Y)Z(p) = [X,[Y,Z]](p)$$

for all $X, Y, Z \in \mathfrak{p}$. In particular, $R(X,Y)X(p) = -(ad_X)^2(Y)(p)$.

Remark 3.3. *We are going to consider only symmetric spaces with nonpositive sectional curvature.*

Fix a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Let Λ denote the set of roots determined by $\mathfrak{a}$, and

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : (adX)w = \alpha(X)w, \forall X \in \mathfrak{a}\}$, $\alpha : \mathfrak{a} \to \mathbb{R}$ is a one-form [J].

Define a corresponding decomposition for each $\alpha \in \Lambda$, $\mathfrak{k}_\alpha = (id + \theta)\mathfrak{g}_\alpha$ and $\mathfrak{p}_\alpha = (id - \theta)\mathfrak{g}_\alpha$. Then:

i. $id + \theta : \mathfrak{g}_\alpha \to \mathfrak{k}_\alpha$ and $id - \theta : \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ are isomorphisms,

ii. $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$, $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$, and $\mathfrak{p}_\alpha \oplus \mathfrak{k}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$,

iii. $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Lambda} \mathfrak{p}_\alpha$, $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Lambda} \mathfrak{k}_\alpha$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$.

For $X \in \mathfrak{a}$ we have that, along the geodesic c in M with initial conditions $c(0) = p$, $c'(0) = X$, the Jacobi fields are linear combinations of the following Jacobi fields:

$$c_{-\alpha(X)^2}(t)v_j(t) \text{ and } s_{-\alpha(X)^2}(t)v_j(t).$$

So, we define for a vector $X \in \mathfrak{a}$, and for α such that $\alpha(X) \neq 0$, the subspaces $P^u_\alpha(X), P^s_\alpha(X) \subset T_{(p,X)}UM$ such that

$$P^u_\alpha(X) = \{(w, |\alpha(X)|w) \in \mathfrak{p} : w \in \mathfrak{p}_\alpha\},$$

$$P^s_\alpha(X) = \{(w, -|\alpha(X)|w) \in \mathfrak{p} : w \in \mathfrak{p}_\alpha\}.$$

If follows from the definition that they are invariant by the geodesic flow.

One can show the following result:

Theorem 3.1. *If the geodesic flow of a compact locally symmetric space of nonpositive curvature is partially hyperbolic, then it is a locally symmetric space of nonconstant negative curvature.*

Demonstração. If the locally symmetric space N has a partially hyperbolic geodesic flow, then the symmetric space M such that $N = M/\Gamma$ has a partially hyperbolic geodesic flow.

Fix $x \in M$ and consider $v \in S_xM$. Let $\mathfrak{a}$ be the maximal Abelian subspace of $\mathfrak{g}$ in x such that $v \in \mathfrak{a}$, after identification of T_xM and $\mathfrak{p}$.

Suppose $dim(\mathfrak{a}) \geq 2$. If the geodesic flow of the symmetric space M is partially hyperbolic, then there is a splitting into invariant subbundles:

$$S(UM) = E^s \oplus E^c \oplus E^u.$$

This decomposition and the curvature tensor formula imply that

$$E^u(x,v) = \{(\xi,\eta) \in T_{(x,v)}UM : (\xi,\eta) \in P^u_{\alpha_i}(v)\},$$

$$E^s(x,v) = \{(\xi,\eta) \in T_{(x,v)}UM : (\xi,\eta) \in P^s_{\alpha_i}(v)\},$$

$i = 1,\dots,k$, $|\alpha_1(v)| > |\alpha_2(v)| > \dots > |\alpha_k(v)|$, such that if $\beta \neq \alpha_i$, $\forall i = 1,\dots,k$, then $\beta(v) < \alpha_i(v)$, $\forall i = 1,\dots,k$.

Now we pick (x, v') such that $\alpha_1(v') = 0$. Then:

$$E^u(x,v') = \{(\xi,\eta) \in T_{(x,v')}UM : (\xi,\eta) \in P_{\beta_j}\},$$

$$E^s(x,v') = \{(\xi,\eta) \in T_{(x,v')}UM : (\xi,\eta) \in P_{\beta_j}\},$$

for some $\beta_j \in \Lambda$, $j = 1,\dots,k'$, $|\beta_1(v')| > |\beta_2(v')| > \dots > |\beta_{k'}(v')|$. Notice that $\alpha_1(v') = 0$ implies $\beta_j \neq \alpha_1$, $\forall j = 1,\dots,k'$. As in the proof of the product metric, there is no way to go from one decomposition to the other continuously. So, there are no Abelian subspaces with dimension greater than one, and the symmetric space of nonpositive curvature has rank one. If dimension of the Abelian subspaces is one then the symmetric space has negative curvature, which implies by the classification of Cartan [H], [He] that it has constant negative curvature or it is a Kähler hyperbolic space, or quaternionic hyperbolic space, or the hyperbolic space over the Cayley numbers. □

4 Locally symmetric manifolds of noncompact type and of rank one

Heintze proved that the following spaces are the only simply connected symmetric manifolds of negative curvature [H]:

i. the hyperbolic space $\mathbb{R}H^n$ of constant curvature $-a^2$, which is the canonical space form of negative constant curvature;

ii. the hyperbolic space $\mathbb{C}H^n$ of curvature $Im(K) \in [-4a^2, -a^2]$, which is the canonical Kähler hyperbolic space of constant negative holomorphic curvature $-4c^2$ [G];

iii. the hyperbolic space $\mathbb{H}H^n$ of curvature $Im(K) \in [-4a^2, -a^2]$, which is the canonical quaternionic Kähler symmetric space of negative curvature [Be], [Wo];

iv. the hyperbolic space CaH^2 of curvature $Im(K) \in [-4a^2, -a^2]$, which is the canonical hyperbolic symmetric space of the octonions of constant negative curvature.

All the cases are hyperbolic, as proved in section 2. They are partially hyperbolic if they have a hyperbolic splitting with more than two invariant subbundles with domination. This is not the case of constant negative curvature: we cannot split the hyperbolic splitting into several invariant subbundles. But we can split the hyperbolic splitting in the case of nonconstant negative curvature.

Locally symmetric spaces with non constant negative curvature have the following decomposition, at $v \in UM$:

$$\mathfrak{p}_0 = \{v\}, a, 2a \text{ are real numbers and } a, 2a \in \Lambda, \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_a \oplus \mathfrak{p}_{2a},$$

so they have the following parallel subspaces of T_xM, for all $x \in M$, $v \in T_xM$:

$$A(x,v) := \{w \in T_xM : K(v,w) = -4a^2\}, \tag{4.2}$$

$$B(x,v) := \{w \in T_xM : K(v,w) = -a^2\}, \tag{4.3}$$

where $a \in \mathbb{R}$ and

$$\mathbb{R}v \oplus A(x,v) \oplus B(x,v) = T_pM.$$

The curvature tensor for locally symmetric manifolds of noncompact type and rank one is

$$R(v,\eta)v = -4a^2\eta_A - a^2\eta_B,$$

where $v \in UM$.

Theorem 4.1. *The geodesic flow of the locally symmetric manifolds of noncompact type and rank one is partially hyperbolic.*

Demonstração. Let b and c be positive real numbers. Let $\mathcal{Q}^{b,c}(\eta,\varsigma) = g(\eta_A,\varsigma_A) - b^2 g(\eta_B,\eta_B) - c^2 g(\varsigma_B,\varsigma_B)$ be a quadractic form. The signature of $\mathcal{Q}^{b,c}$ is $(dimA, dimA + 2dimB)$. The derivative of the quadratic form along the geodesic flow is

$$\begin{aligned}
\frac{d}{dt}\mathcal{Q}^{b,c}(\eta,\varsigma) &= \frac{d}{dt}(g(\eta_A,\varsigma_A) - b^2 g(\eta_B,\eta_B) - c^2 g(\varsigma_B,\varsigma_B)) \\
&= g(\varsigma_A,\varsigma_A) - g(R(v,\eta)v,\eta_A) + g(\eta_{A'},\varsigma_A) + g(\eta_A,\varsigma_{A'}) \\
&- 2b^2 g(\eta_B,\varsigma_B) + 2c^2 g(R(v,\eta)v,\varsigma_B) - 2b^2 g(\eta_{B'},\eta_B) - 2c^2 g(\varsigma_{B'},\varsigma_B) \\
&= g(\varsigma_A,\varsigma_A) - g(R(v,\eta)v,\eta_A) - 2b^2 g(\eta_B,\varsigma_B) + 2c^2 g(R(v,\eta)v,\varsigma_B) \\
&= g(\varsigma_A,\varsigma_A) + 4a^2 g(\eta_A,\eta_A) - 2b^2 g(\eta_B,\varsigma_B) - 2a^2c^2 g(\eta_B,\varsigma_B).
\end{aligned}$$

$$g(\varsigma_A,\varsigma_A)+4a^2g(\eta_A,\eta_A)\geq 4ag(\eta_A,\varsigma_A)\geq 4a(b^2g(\eta_B,\eta_B)+c^2g(\varsigma_B,\varsigma_B))\geq 8abcg(\eta_B,\varsigma_B).$$

We define $e := \frac{b}{c}$. Notice that

$$8abc > 2b^2+2a^2c^2 \Leftrightarrow 3a^2 > e^2-4ae+4a^2 \Leftrightarrow e\in(2a-\sqrt{3}a, 2a+\sqrt{3}a).$$

If $e=2a$, i.e., $b=2ac$, then

$$\frac{d}{dt}\mathcal{Q}^{b,c}(\eta,\varsigma)>0$$

for all $(\eta,\varsigma)\in\overline{\mathcal{C}}_+$.

□

Remark 4.1. *We are free to choose the positive real numbers b and c, but we are bound to a given set of proportions $\frac{b}{c}$. To choose bigger and bigger b and c and do not change the proportion between the two is equivalent to choose 'smaller' cones.*

Remark 4.2. *Since the geodesic flow is partially hyperbolic and hyperbolic at the same time, the splitting is*

$$E(UM)=E^{uu}\oplus E^{cu}\oplus E^{cs}\oplus E^{ss}.$$

The importance of the existence of these kind of splitting is given in [CP], where Pujals and I give an example of a partially hyperbolic geodesic flow which is not hyperbolic, by changing the metric of a compact locally symmetric manifold of nonconstant negative curvature.

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