BLOCK-CACTUS GRAPHS ARE TYPE 1*

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Abstract

The total chromatic number $\chi_T(G)$ of a graph G is the minimum number of colors needed to color properly the edges and vertices of G. A graph G that obeys the Total Color Conjecture is called type 1 if $\chi_T(G) = \Delta(G)+1$ and type 2 if $\chi_T(G) = \Delta(G)+2$ being $\Delta(G)$ the maximum degree of G. In this paper we present new results about the total coloring of a family of graphs, proving that the nontrivial block-cactus graphs are type 1.

Resumo

O número cromático total $\chi_T(G)$ de um grafo G é o menor número de cores necessário para colorir propriamente os vértices e arestas de G. Um grafo que satisfaz a Conjectura da Coloração Total é dito de tipo 1 se $\chi_T(G) = \Delta(G)+1$ e é tipo 2 se $\chi_T(G) = \Delta(G)+2$, onde $\Delta(G)$ é o grau máximo de G. Neste trabalho, apresentamos novos resultados sobre a coloração total de uma família de grafos, provando que os grafos block-cactus não triviais são de tipo 1.

1. Introduction

Let $G(V, E)$ be a graph and C a finite set of colors. A *total coloring* of $G(V, E)$ is an assignment of colors to its vertices and edges so that adjacent or incident elements have distinct colors. The least number of colors sufficient for a total coloring of a graph is called its *total chromatic number* and it is denoted by $\chi_T(G)$. Clearly, for any graph G, $\Delta(G)+1 \le \chi_T(G)$,

* *Keywords:* total coloring, block-cactus graphs. **AMS Classification**: 05C15

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where $\Delta(G)$ is the maximum degree of G. The *Total Coloring Conjecture* (TCC), posed by Vizing and Behzad independently [15], says:

For any simple graph G, $\chi_T(G) \le \Delta(G)+2$.

The truth of the TCC would imply that $\chi_T(G)$ attains one of two values for any graph G. A graph G is *type 1* if $\chi_T(G) = \Delta(G)+1$ and it is *type 2* if $\chi_T(G) = \Delta(G)+2$.

In recent years, the TCC has been showed for some families of graphs such as interval graphs [1], power of cycles [3] and many cases of planar graphs. But the problem of determining if a graph of a family is type 1 or type 2 is not so easy. And more, sometimes there are examples of the two types belonging to the same family as cycles, complete and bipartite graphs [15]. Sharebaf [12] showed that spider graphs $S_{m,n}$ are type 1 if $m \neq 3,4$, and the other cases are type 2. Planar graphs with $\Delta(G)=6$ without 4-cycles are type 1 [13]. The same classification is possible for planar graphs with $\Delta(G)=8$ with restrictions [4, 9, 11, 14] and also for planar graphs with large maximum degrees such as $\Delta(G)=9$ [10], $\Delta(G)=10$ [16] and $\Delta(G)=11$ [2]. In this paper we present new results about the total coloring of the nontrivial block-cactus graphs. In [5], it was already proved that these graphs obey the TCC and now we show that this family is type 1.

2. Basic Concepts

Let $G(V, E)$ be a simple connected graph, $|V| = n$, the *order* of G, and $|E| = m$, the *size* of G. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V;\ uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of v is $d(v) = |N(v)|$. A vertex $v \in V$ is called *universal* if $N[v] = V$. The *complement* of a graph $G(V,E)$ is the graph $\overline{G}$ with the same vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A complete graph is denoted by K_n and a cycle graph by C_n.

A *cut-vertex* is a vertex that if removed produces a graph with more connected components than the original graph. A *block* of a graph is a maximal connected subgraph containing no cut-vertices. The next theorem was proposed by Harary [6].

Theorem 1. *The intersection of any two distinct blocks of a graph consists of at most one vertex.*

A graph is a *block graph* if it is connected and every block is a clique. A graph is a *cactus graph* if it is connected and every block is a cycle or K_2.

A graph G is a *block-cactus graph* if every block is a clique or a cycle. A *bleaf* of a block-cactus graph is a block having one sole cut-vertex. A *nontrivial* block-cactus graph has at least two bleafs.

Two classic results about total coloring of complete graphs and cycles with n vertices stated that [15]:

$$\chi_T(K_n)=\begin{cases}\Delta(K_n)+1=n, & n \text{ odd}\\ \Delta(K_n)+2=n+1, & n \text{ even;}\end{cases}$$

and

$$\chi_T(C_n)=\begin{cases}\Delta(C_n)+1=3, & n\equiv 0 \bmod 3\\ \Delta(C_n)+2=4, & \text{otherwise.}\end{cases}$$

Some results are found in the literature related to the total coloring of graphs having universal vertices, which are used latter in this work [7,8]. Hilton [7] presented an important one, the complete classification of graphs G of order $2k$, for any positive integer k, with $\Delta(G)=2k-1$, proving the following theorem.

Theorem 2. *Let J be a subgraph of K_{2k}, let $e=|E(J)|$ and let j be the maximum size of a matching in J. Then $\chi_T(K_{2k}\setminus E(J))=2k+1$ if and only if $e+j\le k-1$.*

3. Total Coloring of Block-Cactus Graphs

In this section, it will be proved that nontrivial block-cactus graphs are type 1. In order to achieve this result, a simple lemma must be shown.

Lemma 3. *Let $G(V,E)$ be a block-cactus graph with exactly two blocks. Then G is type 1.*

Proof. Observe that G has only one cut-vertex v. We will consider two cases.

- Case 1: The blocks are both cliques.

  In this case, the vertex v is an universal vertex.

  If $|V|$ is odd and since G is subgraph of K_n and $\chi_T(K_n)=\Delta(K_n)+1$, then G is type 1.

  Otherwise, G has even order and by Theorem 2 it is necessary to analyze $\overline{G}$ in order to prove that $e+j\ge k$.

  Let B_1 and B_2 be the cliques of G such that $x=|V(B_1)|-1$, $y=|V(B_2)|-1$; consequently $|V(G)|=x+y+1$. Thus, $\overline{G}$ is composed by two connected subgraphs, the complete bipartite graph $K_{x,y}$ and the isolated vertex $\{v\}$.

  Without loss of generality, let $x\le y$. So, considering the notation of Theorem 2 we have $e+j=xy+x$ and, since $2k=x+y+1$, $k=\dfrac{x+y+1}{2}$.

  It is easy to see that $xy+x\ge\dfrac{x+y+1}{2}$. We do not have $e+j\le k-1$. Then, G is type 1.

- Case 2: At least one block is a cycle.

  In this case, $\Delta(G) = d(v) \geq 3$. A cycle is totally colored with three or four colors. Firstly, we color v and its incident edges in G with $\Delta(G)+1$ colors.

  Let B_1 and B_2 be cycles. B_1 can be colored with the color of v, the two colors already used in the two incident edges of v and, if necessary, one color of an incident edge of v belonging to B_2. The same approach is used to color B_2.

  Now, let B_1 be a cycle and B_2 be a clique. We color B_1 as suggested before. If B_2 is clique of even order t, then we can use the color of v, the $(t-1)$ colors of the incident edges in v belonging to B_2 and one color of an incident edge of v belonging to B_1 to complete the coloring of B_2. If B_2 is a clique of odd order t, we use only the color of v and the $(t-1)$ colors of the incident edges in v belonging to B_2 to color the others vertices and edges of B_2.

  Then, G is type 1. ■

***Theorem 4**. A nontrivial block-cactus graph $G(V, E)$ is type 1.*

Proof. The proof is by induction on the number of blocks.

If G is composed of two blocks then, by Lemma 3, G is type 1.

Otherwise, let us consider a bleaf $B(V(B),E(B))$ of G; B is a clique K_t or a cycle C_t.

Let v_1 be the unique cut-vertex of G that belongs to B, being $N[v_1] = \{v_1, v_2, ..., v_{d_G(v_1)+1}\}$.

If $B = K_t$ then $k = d_G(v_1)+1-t$, otherwise $k = d_G(v_1)-2$. In any case, $V(B) = \{v_1, v_{k+2}, ..., v_{d_G(v_1)+1}\}$.

Let us remove the vertices of the set $V(B)\setminus\{v_1\}$ from G and let $G'(V', E')$ be the induced subgraph of G resulting from this operation. Observe that $k = d_{G'}(v_1)$.

By the induction hypothesis, the block-cactus graph G' is type 1, that is, it has a total coloring

$$c'_T : V' \cup E' \rightarrow C' = \{c_1, c_2, ..., c_{\Delta(G')+1}\}.$$

Let us now add the block B to the graph G', already colored; the result is the graph G. We must extend the coloring c'_T in order to obtain the desired total coloring of G.

The vertex v_1 and its incident edges in G' are colored, without loss of generality, with the colors $c_1, c_2, ..., c_{k+1}$, respectively. The set of colors $C^* = \{c_{k+2}, c_{k+3}, ..., c_{d_G(v_1)+1}\}$ is available for the coloring of the block B. We observe that $|C^*| = t-1$ if $B = K_t$, otherwise $|C^*| = 2$. We must analyze two cases in order to prove that there is at least one color of C', not belonging to C^* and different from c_1, to extend this coloring.

- $\Delta(G') < \Delta(G)$

  So, $d_G(v_1) = \Delta(G) = d_{G'}(v_1) + d_B(v_1) = \begin{cases} k+t-1, & B = K_t \\ k+2, & B = C_t. \end{cases}$

  The graph G' is composed of at least two blocks, then $|C'| \geq 3$.

  In the worst case, $k = 1$. So, the set $C^* = \{c_3, c_4, \ldots, c_{\Delta(G)+1}\}$ and the color c_2 can be used in block B. Otherwise, more colors will be available.

  Furthermore, the set $C' \cup C^*$ has $\Delta(G)+1$ colors.

- $\Delta(G') = \Delta(G)$

  In this case, $C^* \subset C'$ and $\{c_1, c_2, \ldots, c_{k+1}\} \subset C'$, but $\{c_1, c_2, \ldots, c_{k+1}\} \not\subset C^*$.

  So, besides the colors of C^*, there are at least k colors (not considering c_1) available to color the block B.

  Finally, $C' \cup C^* = C'$ having $\Delta(G)+1$ colors.

Then, in any case, the block-cactus graph G is type 1. ■

In Theorem 4, it was proved that nontrivial block-cactus graphs are type 1. There are different approaches to this task. We have elected to present a more detailed proof in order to color the graph without needing to reassign the colors. Algorithms that reassign colors have to pass through the vertices and edges several times; so, the chosen approach has an important application: providing an efficient algorithm for total coloring the graph with $\Delta(G)+1$ colors.

4. Conclusion

In this paper, it was shown that all nontrivial block-cactus graphs are type 1. It is interesting to note that this family actually includes two subfamilies: block graphs and cactus graphs. So, as cactus graphs are also planar, we are also solving here same particular cases for this family. On the other hand, block graphs are chordal, and a question we may leave open is to determine other subclasses of chordal graphs for which it can be proved that all members are type 1.

Acknowledgement

This work was supported by grant 305372/2009-2, CNPq, Brazil.

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