On Exact Diagrams of Linear Mappings between Spaces of Formal Series*

CLÁUDIA F. R. CONCORDIDO † DINAMÉRICO P. POMBO JR. ‡

Abstract

The exactness of certain diagrams of linear mappings between vector spaces of formal series (resp. continuous polynomials) is established.

A fundamental result of Linear Algebra([4], Chap.III, §2, Proposition 2.1) asserts that exact sequences of linear mappings between modules give raise to exact sequences of group homomorphisms between additive groups of linear mappings. In the same vein, it was shown in Theorem IV.4 of [7] that exact diagrams (in the sense of Grothendieck) of linear mappings between modules give raise to exact diagrams of linear mappings between modules of **lois polynomes**. The main purpose of this paper is to establish a continuous version of the last mentioned result. More precisely, we shall prove that the exactness of certain diagrams of continuous linear mappings between topological vector spaces implies the exactness of certain diagrams of linear mappings between vector spaces of formal series and the exactness of certain diagrams of linear mappings between vector spaces of continuous polynomials.

Let us begin with some preliminaries.

Definition 1 [2],[3] *A diagram*

$$A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \xrightarrow{g} C \ \ (\ resp.\ \ A \xrightarrow{f} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C\ )$$

of mappings between sets is said to be exact if g is surjective and if for each $y_1,\, y_2\, \in B$ the following conditions are equivalent:
$g(y_1) = g(y_2)$; there exists an $x \in A$ such that $y_1 = f_1(x)$ and $y_2 = f_2(x)$
(resp. f is a bijection from A onto the set $\{y \in B\, ;\, g_1(y) = g_2(y)\}$).

Throughout this paper $I\!\!K$ shall denote a separated non-discrete topological field of characteristic zero.

Example 2 *Let E, F and G be three topological vector spaces over $I\!\!K$, and let $\alpha : E \longrightarrow F$ and $w : F \longrightarrow G$ be continuous linear mappings. Consider $E \times F$ endowed with the product topology and define $u, v : E \times F \longrightarrow F$ by $u(x,y) = y$ and $v(x,y) = \alpha(x) + y$ (u and v are continuous linear mappings). If the sequence*

$$0 \longrightarrow E \xrightarrow{\alpha} F \xrightarrow{w} G \longrightarrow 0$$

is exact, then it is easily verified that the diagram

* *Palavras chave*: topological vector spaces, formal series, continuous polynomials, exact diagrams.
† Instituto de Matemática e Estatística, UERJ, concordido@ime.uerj.br
‡ Instituto de Matemática, UFF, dpombo@terra.com.br

$$E \times F \underset{v}{\overset{u}{\rightrightarrows}} F \xrightarrow{w} G$$

is exact.

In particular, let E be a topological vector space over $I\!K$, M a vector subspace of E and $\pi : E \longrightarrow E/M$ the canonical surjection. Consider M endowed with the induced topology, $M \times E$ endowed with the product topo-logy and E/M endowed with the quotient topology. Then the diagram (of continuous linear mappings)

$$M \times E \underset{v}{\overset{u}{\rightrightarrows}} E \xrightarrow{\pi} E/M$$

is exact, where $u(x,y) = y$ and $v(x,y) = x + y$ for $(x,y) \in M \times E$. Moreover, π is open.

Definition 3 [6] *Let m be a positive integer, and let E and F be two topological vector spaces over $I\!K$. A mapping $P : E \longrightarrow F$ is said to be an m-homogeneous polynomial if there exists a symmetric m-linear mapping $A : E^m \longrightarrow F$ such that $P(x) = A(\underbrace{x, \ldots, x}_{m\ times})$ for all $x \in E$. We shall write $P = \hat{A}$ to indicate that P corresponds to A in this way.*

The following fundamental "Polarization Formula" is well-known [1]:

If $A : E^m \longrightarrow F$ is a symmetric m-linear mapping, then

$$A(x_1, \ldots, x_m) = \frac{1}{(m!)e} \sum_{\epsilon_1, \ldots, \epsilon_m \in \{0,1\}} (-e)^{m-(\epsilon_1 + \ldots + \epsilon_m)}\ \hat{A}((\epsilon_1 e)x_1 + \ldots + (\epsilon_m e)x_m)$$

for all $(x_1, \ldots, x_m) \in E_1 \times \ldots \times E_m$, where e is the identity element of $I\!K$.

It follows from this formula that for each m-homogeneous polynomial $P : E \longrightarrow F$ there exists a unique symmetric m-linear mapping $A : E^m \longrightarrow F$ such that $\hat{A} = P$.

The next result is an immediate consequence of Theorem 41 of [1].

Proposition 4 *Let E and F be two topological vector spaces over $I\!K$, $P : E \longrightarrow F$ an m-homogeneous polynomial and $A : E^m \longrightarrow F$ the symmetric m-linear mapping such that $P = \hat{A}$. Then the following conditions are equivalent:*

(a) P is continuous;

(b) P is continuous at $0 \in E$;

(c) A is continuous;

(d) A is continuous at $(0, \ldots, 0) \in E^m$.

We shall denote by $\mathcal{L}^m(E,F)$ the vector space over $I\!K$ of all continuous m-homogeneous polynomials from E into F, and we shall define $\mathcal{L}^0(E,F) = F$.

Definition 5 [6] *Given two topological vector spaces E and F over $I\!K$, the product vector space*

$$\prod_{m\in I\!N}\mathcal{L}^m(E,F)$$

shall be represented by $F\,[[E]]$; an element of $F\,[[E]]$ is called a formal series from E into F. The vector subspace $\bigoplus_{m\in I\!N}\mathcal{L}^m(E,F)$ of $F\,[[E]]$ shall be represented by $F[E]$; an element of $F[E]$ is called a continuous polynomial from E into F.

Let E and F be two topological vector spaces over $I\!K$ and $u : E \longrightarrow F$ a continuous linear mapping. For each topological vector space H over $I\!K$, let u_H be the linear mapping from $H\,[[F]]$ into $H\,[[E]]$ given by

$$u_H\left((P_m)_{m\in IN}\right) = (P_m \circ u)_{m\in IN}$$

(if $P_0 = z \in \mathcal{L}^0(F,H)$, $P_0 \circ u$ means z). Obviously, $u_H|_{(H[F])}$ is a linear mapping from $H[F]$ into $H[E]$.

We can now state our main result:

Theorem 6 *Let E, F and G be three topological vector spaces over $I\!K$, and let $u, v : E \longrightarrow F$ and $w : F \longrightarrow G$ be continuous linear mappings. Assume that the diagram*

$$E \underset{v}{\overset{u}{\rightrightarrows}} F \xrightarrow{w} G$$

is exact and that w is an open mapping. Then, for each topological vector space H over $I\!K$, the diagrams

$$H[[G]] \xrightarrow{w_H} H[[F]] \underset{v_H}{\overset{u_H}{\rightrightarrows}} H[[E]]$$

and

$$H[G] \xrightarrow{w_H|_{(H[G])}} H[F] \underset{v_H|_{(H[F])}}{\overset{u_H|_{(H[F])}}{\rightrightarrows}} H[E]$$

are exact.

Proof. Let us prove the first assertion. Indeed, let H be an arbitrary topological vector space over $I\!K$. We claim that w_H is injective. In fact, let $(P_m)_{m\in IN} \in H[[G]]$ be such that $w_H((P_m)_{m\in I\!N}) = (P_m \circ w)_{m\in I\!N} = 0$. Then $P_0 = 0$ and the surjectivity of w implies that $P_m = 0$ for all positive integer m. Thus w_H is injective.

Now, we claim that

$$Im(w_H) = \left\{(Q_m)_{m\in I\!N} \in H[[F]];\ u_H((Q_m)_{m\in I\!N}) = v_H((Q_m)_{m\in I\!N})\right\}.$$

In fact, let $(Q_m)_{m\in I\!N} \in Im(w_H)$. Then there exists a $(P_m)_{m\in I\!N} \in H[[G]]$ such that $P_m \circ w = Q_m$ for all $m \in I\!N$. But, since $w \circ u = w \circ v$, then

$$Q_m \circ u = (P_m \circ w) \circ u = P_m \circ (w \circ u) = P_m \circ (w \circ v) = (P_m \circ w) \circ v = Q_m \circ v$$

for all $m \in I\!N$. Therefore $u_H((Q_m)_{m\in I\!N}) = v_H((Q_m)_{m\in I\!N})$.

In order to prove the other inclusion, let us first verify that if m is a positive integer and $Q \in \mathcal{L}^m(F,H)$ is such that $Q\circ u = Q\circ v$, then there exists a $P \in \mathcal{L}^m(G,H)$ such that $P\circ w = Q$. In fact, let $B : F^m \longrightarrow H$ be the continuous symmetric m-linear mapping such that $\widehat{B} = Q$ (Proposition 4). If $z_1,\dots,z_m \in G$ are arbitrary, let $y_1, y_1', \dots, y_m, y_m' \in F$ be such that $w(y_i) = w(y_i') = z_i$ for $i = 1,\dots,m$. By hypothesis, for each $i = 1,\dots,m$ there exists an $x_i \in E$ such that $u(x_i) = y_i$ and $v(x_i) = y_i'$. Consequently, if $\epsilon_1,\dots,\epsilon_m \in \{0,1\}$,

$$Q(\,(\epsilon_1 e)y_1 + \ldots + (\epsilon_m e)y_m\,) = Q(\,(\epsilon_1 e)u(x_1) + \ldots + (\epsilon_m e)u(x_m)\,)$$

$$= Q(\,u((\epsilon_1 e)x_1 + \ldots + (\epsilon_m e)x_m)\,) = (Q\circ u)(\,(\epsilon_1 e)x_1 + \ldots + (\epsilon_m e)x_m\,)$$

$$= (Q\circ v)(\,(\epsilon_1 e)x_1 + \ldots + (\epsilon_m e)x_m\,) = Q(\,v((\epsilon_1 e)x_1 + \ldots + (\epsilon_m e)x_m)\,)$$

$$= Q\,(\,(\epsilon_1 e)v(x_1) + \ldots + (\epsilon_m e)v(x_m)\,) = Q\,(\,(\epsilon_1 e)y_1' + \ldots + (\epsilon_m e)y_m'\,),$$

and the Polarization Formula furnishes $B(y_1,\dots,y_m) = B(y_1',\dots,y_m')$. Define $A : G^m \longrightarrow H$ by $A(z_1,\dots,z_m) = B(y_1,\dots,y_m)$, where $y_i \in F$ and $w(y_i) = z_i$ for $i = 1,\dots m$. By what we have just seen, A is well-defined, and it is clear that A is a symmetric m-linear mapping. We claim that the m-homogeneous polynomial $P{=}\hat{A}$ is continuous. Indeed, let W be a neighborhood of 0 in H. By the continuity of B at $(0,\dots,0) \in F^m$, there exists a neighborhood V of 0 in F such that $B\left(\underbrace{V\times\ldots\times V}_{m\ \text{times}}\right) \subset W$. Since, by hypothesis, w is surjective and open, $w(V)$ is a neighborhood of 0 in G; and, by the definition of A, $A\left(\underbrace{w(V)\times\ldots\times w(V)}_{m\ \text{times}}\right) \subset W$. Thus A is continuous at $(0,\dots,0) \in G^m$, and hence $P \in \mathcal{L}^m(G,H)$ by Proposition 4. Moreover,

$$(P\circ w)(y) = P(w(y)) = A(w(y),\dots,w(y)) = B(y,\dots,y) = Q(y)$$

for all $y \in F$, that is, $P\circ w = Q$.

Now, let $(Q_m)_{m\in I\!N} \in H[[F]]$ be such that $u_H((Q_m)_{m\in I\!N}) = v_H((Q_m)_{m\in I\!N})$. Then $Q_m\circ u = Q_m\circ v$ for all $m \in I\!N$, and by what we have just seen for each positive integer m there exists a $P_m \in \mathcal{L}^m(G,H)$ such that $P_m\circ w = Q_m$. Put $P_0 = Q_0$. Then $w_H((P_m)_{m\in I\!N}) = (Q_m)_{m\in I\!N}$, and the proof of the first assertion is complete.

Let us prove the second assertion. Firstly, $w_H|_{(H[G])}$ is obviously injective. Let $(Q_m)_{m\in I\!N} \in Im(w_H|_{(H[G])})$. Then there exists a $(P_m)_{m\in I\!N} \in H[G]$ such that $w_H((P_m)_{m\in I\!N}) = (Q_m)_{m\in I\!N}$. Hence $u_H((Q_m)_{m\in I\!N}) = v_H((Q_m)_{m\in I\!N})$, as we have just verified. Finally, let $(Q_m)_{m\in I\!N} \in H[F]$ be such that $u_H((Q_m)_{m\in I\!N}) = v_H((Q_m)_{m\in I\!N})$, and let l be a positive integer such that $Q_m = 0$ for $m \geq l$. As we have seen in the proof of the first assertion, for each $m = 0,\dots,l-1$ there is a $P_m \in \mathcal{L}^m(G,H)$ such that $P_m\circ w = Q_m$. Define $(R_m)_{m\in I\!N} \in H[G]$ by $R_m = P_m$ for $m = 0,\dots,l-1$ and $R_m = 0$ for $m \geq l$. Then $w_H((R_m)_{m\in I\!N}) = (Q_m)_{m\in I\!N}$. This completes the proof of the theorem.

The next example shows that the condition that w is open is essential for the validity of Theorem 6.

Example 7 *Let E be an infinite-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$, G the vector space E endowed with the weak topology $\sigma(E, E')$ and 1_E the identity mapping of E. Then*

$$E \underset{1_E}{\overset{1_E}{\rightrightarrows}} E \xrightarrow{w} G$$

is an exact diagram of continuous linear mappings, where $w = 1_E$. Moreover, w is not open. On the other hand, the diagrams

$$E[[G]] \xrightarrow{w_E} E[[E]] \underset{(1_E)_E}{\overset{(1_E)_E}{\rightrightarrows}} E[[E]]$$

and

$$E[G] \xrightarrow{w_E|(E[G])} E[E] \underset{(1_E)_E|(E[E])}{\overset{(1_E)_E|(E[E])}{\rightrightarrows}} E[E]$$

are not exact. In fact, if $(Q_m)_{m\in\mathbb{N}}$ is the element of $E[E]$ given by $Q_m = 0$ for $m \neq 1$ and $Q_1 = 1_E : E \longrightarrow E$, then the only mapping $P_1 : E \longrightarrow E$ satisfying $P_1 \circ w = Q_1$ is the identity mapping of E, which is not continuous as a mapping from G into E.

Corollary 8 *Let E be a topological vector space over $\mathbb{K}$, M a vector subspace of E and $\pi : E \longrightarrow E/M$ the canonical surjection. Consider M endowed with the induced topology, $M \times E$ endowed with the product topology and E/M endowed with the quotient topology, and let $u, v : M \times E \longrightarrow E$ be the continuous linear mappings given by $u(x, y) = y$ and $v(x, y) = x + y$. Then, for each topological vector space H over $\mathbb{K}$, the diagrams*

$$H[[E/M]] \xrightarrow{\pi_H} H[[E]] \underset{v_H}{\overset{u_H}{\rightrightarrows}} H[[M \times E]]$$

and

$$H[E/M] \xrightarrow{\pi_H|(H[E/M])} H[E] \underset{v_H|(H[E])}{\overset{u_H|(H[E])}{\rightrightarrows}} H[M \times E]$$

are exact.

Proof. In view of Example 2, the corollary is an immediate consequence of Theorem 6.

Corollary 9 *Let E, F and G be three topological vector spaces over a non-trivially valued field L of characteristic zero such that F is metrizable and complete and G is separated and barrelled ([5], Definition 2.35). Let $u, v : E \longrightarrow F$ and $w : F \longrightarrow G$ be continuous linear mappings. Assume that the diagram*

$$E \underset{v}{\overset{u}{\rightrightarrows}} F \xrightarrow{w} G$$

is exact. Then, for each topological vector space H over L, the diagrams

$$H[[G]] \xrightarrow{w_H} H[[F]] \overset{u_H}{\underset{v_H}{\rightrightarrows}} H[[E]]$$

and

$$H[G] \xrightarrow{w_H|(H[G])} H[F] \overset{u_H|(H[F])}{\underset{v_H|(H[F])}{\rightrightarrows}} H[E]$$

are exact.

Proof. By Theorem 2.73 of [5], w is an open mapping. Therefore the corollary follows immediately from Theorem 6.

Corollary 10 *Let F and G be two topological vector spaces over a non-trivially valued field L of characteristic zero such that F is metrizable and complete and G is separated and barrelled, and let $w : F \longrightarrow G$ be a continuous surjective linear mapping. Consider $Ker(w)$ endowed with the induced topology and $Ker(w) \times F$ endowed with the product topology, and let $u, v : Ker(w) \times F \longrightarrow F$ be the continuous linear mappings given by $u(x, y) = y$ and $v(x, y) = x + y$. Then, for each topological vector space H over L, the diagrams*

$$H[[G]] \xrightarrow{w_H} H[[F]] \overset{u_H}{\underset{v_H}{\rightrightarrows}} H[[Ker(w) \times F]]$$

and

$$H[G] \xrightarrow{w_H|(H[F])} H[F] \overset{u_H|(H[F])}{\underset{v_H|(H[F])}{\rightrightarrows}} H[Ker(w) \times F]$$

are exact.

Proof. Since, by Example 2, the diagram

$$Ker(w) \times F \overset{u}{\underset{v}{\rightrightarrows}} F \xrightarrow{w} G$$

is exact, the result follows immediately from Corollary 9.

By Theorem 2.37 of [5], Corollaries 9 and 10 hold if G is a separated topological vector space over L which is a Baire space, and hence if G is a metrizable and complete topological vector space over L.

References

[1] C.F.R. Concordido & D.P. Pombo Jr., *Polynomials and multilinear mappings in topological modules*, Rend. Circ. Mat. Palermo 51 (2002), 213-236.

[2] A. Grothendieck, *Technique de Descente et Théorèmes d'Existence en Géométrie Algébrique, I: Généralités. Descent par Morphismes Fidèlement Plats*, Séminaire Bourbaki 1959/1960, exposé 190.

[3] A. Grothendieck & J. A. Dieudonné, *Eléments de Géométrie Algébrique I*, Grundlehren der mathematischen Wissenschaften 166, Springer-Verlag (1971).

[4] S. Lang, *Algebra*, Third Edition, Addison-Wesley (1993).

[5] J.B. Prolla, *Topics in Functional Analysis over Valued Division Rings*, Notas de Matemática 89, North-Holland (1982).

[6] J.-P. Ramis, *Sous-ensembles analytiques d'une variété banachique complexe*, Ergebnisse der Mathematik und ihrer Grenzgebiete 53, Springer-Verlag (1970).

[7] N. Roby, *Lois polynomes et lois formelles en théorie des modules*, Ann. Sci. École Norm. Sup. 80 (1963), 213-348.