Hidden Regularity for a Hyperbolic Equation with a Resistance Term *

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Abstract

In this paper we obtain the hidden regularity for the weak solution of the nonlinear equation whose linear model is originated from a problem of elasticity with a resistance of the material.

Resumo

Neste artigo obtemos a Regularidade Escondida para a solução fraca de uma equação não-linear, cujo modelo tem origem em um problema de elasticidade com uma resistência do material.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary Γ. We represent by Q the cylinder $\Omega\times(0,T)$ with lateral boundary $\Sigma=\Gamma\times(0,T)$, T an arbitrary positive real number. In this article we consider the system

$$\left|\begin{array}{l} u''-\Delta u+F(u)=-\nabla p \quad \text{in} \quad Q,\\ div\ u=0 \quad \text{in} \quad Q,\\ u=0 \quad \text{on} \quad \Sigma,\\ u(0)=u_0, \quad u'(0)=u_1 \quad \text{in} \quad \Omega, \end{array}\right. \tag{1.1}$$

that appear in the theory of elasticity of incompressible material. We observe that $u:Q\longrightarrow\mathbb{R}^n$, $u=(u_1,u_2,...,u_n)$, where $u_i:Q\longrightarrow\mathbb{R}$ for $i=1,...,n$, $div\ u=\sum\limits_{i=1}^{n}\dfrac{\partial u_i}{\partial x_i}$, $F:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is given by

$$F(u)=\begin{bmatrix} f_1 & 0 & ... & 0\\ 0 & f_2 & ... & 0\\ \vdots & & & \vdots\\ 0 & 0 & ... & f_n \end{bmatrix}\begin{bmatrix} u_1\\ u_2\\ \vdots\\ u_n\end{bmatrix},$$

where we assume that each $f_i:\mathbb{R}\longrightarrow\mathbb{R}$, $i=1,...,n$ is a continuous function that satisfies

$$f'\in L^\infty(\mathbb{R}), \qquad \lim_{s\to\infty}\frac{f_i(s)}{s}=\alpha_i,\ i=1,...,n. \tag{1.2}$$

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This model, without the nonlinear term, was proposed in the first time in a work of J. L. Lions [6]. The aim of this paper it shows that, under the condition (1.2) and a geometrical condition on Ω, the weak solution of the problem (1.1) has the following regularity on the lateral boundary Σ

$$\frac{\partial u}{\partial \nu} \in \left[L^2(\Sigma)\right]^n, \tag{1.3}$$

where ν is the exterior normal vector to Γ. The property (1.3) was proved by J. L. Lions [6], motivated by problems of optimal control for the equation $u'' - \Delta u = 0$ in Q. In [7], J. L. Lions proves that the weak solution u of the mixed problem for the equation

$$u'' - \Delta u + |u|^\rho u = 0 \text{ in } Q,$$

also satisfies (1.3) and called this property of "*Hidden Regularity*", because that one does not follow from the equation or from the spaces where the solution lies. This result was generalized by M. Milla Miranda and L. A. Medeiros [8] for the equation

$$u'' - \Delta u + F(u) = 0 \text{ in } Q,$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the hypothesis of W. A. Strauss [10], i.e., F is continuous and $sF(s) \geq 0$, $s \in \mathbb{R}$. In [11], M. C. de Campos Vieira obtain the Hidden Regularity for the following Coupled System of Nonlinear Hyperbolic Equations

$$\begin{aligned} u'' - \Delta u + \alpha |v|^{\rho+2} |u|^\rho u &= g_1 \text{ in } Q \\ v'' - \Delta v + \alpha |u|^{\rho+2} |v|^\rho v &= g_2 \text{ in } Q. \end{aligned}$$

The *Hidden Regularity* for the linear system associated to (1.1) was proved in [5] by Lions. For a physical interpretation of this model see Rocha dos Santos [9]. See about similar questions in Kapitonov [3], [4].

2 Existence and Uniqueness of Solution

In this section we study the existence and uniqueness of solution for the problem (1.1) that will be written in the following form:

$$\left| \begin{array}{l} u_i'' - \Delta u_i + f_i(u_i) = -\dfrac{\partial p}{\partial x_i} \quad \text{in} \quad Q, \\ \sum\limits_{j=1}^{n} \dfrac{\partial u_i}{\partial x_j} = 0 \quad \text{in} \quad Q, \\ u_i = 0 \quad \text{on} \quad \Sigma, \\ u_i(0) = u_{0i}, \quad u_i'(0) = u_{1i} \quad \text{in} \quad \Omega, \end{array} \right. \tag{2.4}$$

for $i = 1, ..., n$, where the prime means the derivative with respect to t and all the derivatives are in the sense of the theory of distributions.

Before beginning the study of (2.4), the notation will be established. We consider the Hilbert spaces

$$V = \left\{ u \in \left(H_0^1(\Omega)\right)^n ;\ div\ u = 0 \right\}$$

and

$$H = \left\{ u \in \left(L^2(\Omega)\right)^n ;\ div\ u = 0 \text{ and } u.\nu = 0 \text{ on } \Gamma \right\},$$

equipped with the inner product and norm given respectively by

$$((u,v)) = \sum_{i=1}^{n} ((u_i, v_i))_{H_0^1(\Omega)}, \qquad \|u\|^2 = \sum_{i=1}^{n} \|u_i\|^2_{H_0^1(\Omega)}$$

and

$$(u,v) = \sum_{i=1}^{n} (u_i, v_i)_{L^2(\Omega)}, \qquad |u|^2 = \sum_{i=1}^{n} |u_i|^2_{L^2(\Omega)}.$$

Remark 2.1 *Note that it is necessary to show that for each* $w \in H_0^1(\Omega)$, $f_i(w) \in L^2(\Omega)$*. In fact, from the hypothesis* (1.2) *follows that given* $\varepsilon = 1$ *there exists* $N_i = N_i(\varepsilon)$ *such that*

$$|f_i(w) - \alpha_i w| < |w|, \ \forall \ |w| > N_i,$$

this is,

$$|f_i(w)|^2 < 2\left(|\alpha_i w|^2 + |w|^2\right), \ \ |w| > N_i. \tag{2.5}$$

From (2.5) *we obtain*

$$\int_\Omega |f_i(w)|^2 \, d\Omega < \int_{|w| \le N_i} |f_i(w)|^2 \, d\Omega + \int_{|w| > N_i} |f_i(w)|^2 \, d\Omega <$$
$$< C_i + 2\left(|\alpha_i|^2 + 1\right) \int_\Omega |w|^2 \, d\Omega,$$

because f_i *is continuous and* Ω *is bounded.*

Theorem 2.1 *Given* $u_0 \in V$, $u_1 \in H$ *and* f_i *satisfying* (1.2), *there exists a unique function* u *such that*

$$u \in L^{\infty}(0,T;V), \tag{2.6}$$

$$u' \in L^{\infty}(0,T;H), \tag{2.7}$$

$$u'' - \Delta u + F(u) = -\nabla p \quad in \quad \mathcal{D}'\left(0,T;(\mathcal{D}'(\Omega))^3\right), \tag{2.8}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad in \quad \Omega. \tag{2.9}$$

Before beginning the proof of the Theorem 2.1, let us consider the following lemma:

Lemma 2.1 *Considering* u_0, u_1 *and* f_i *in the conditions of the Theorem 3.1. Then there exists a unique function* u *satisfying*

$$u \in L^{\infty}(0,T;V), \tag{2.10}$$

$$u' \in L^{\infty}(0,T;H), \tag{2.11}$$

$$\frac{d}{dt}(u'(t),v) + ((u(t),v)) + (F(u),v) = 0, \ \forall v \in V \quad in \quad \mathcal{D}'(0,T), \tag{2.12}$$

$$u(0) = u_0, \quad u'(0) = u_1 \quad in \quad \Omega. \tag{2.13}$$

Proof. We employ the Faedo-Galerkin's method. Let $\{w_\nu\}_{\nu\in\mathbb{N}}$ a Hilbertian basis of V. We consider $V_m = [w_1,...,w_m]$ the subspace of V generated by the m first vectors of $\{w_\nu\}_{\nu\in\mathbb{N}}$. The approximate problem of (2.12) – (2.13) consists in to find $u_m(t) \in V_m$ satisfying for each $i = 1,...,n$

$$\left| \begin{array}{l} \dfrac{d}{dt}(u'_{im}(t),v) + ((u_{im}(t),v)) + (f_i(u_{im}(t)),v) = 0, \ \forall v \in V_m, \\ u_{im}(0) = u_{0im} \to u_{0i} \text{ strong in } H_0^1(\Omega), \\ u'_{im}(0) = u_{1im} \to u_{1i} \text{ strong in } L^2(\Omega). \end{array} \right. \tag{2.14}$$

By Carathèodory's theorem, (2.14) has solution on an interval $[0,t_m[$, $t_m < T$. This solutions extend to the whole interval $[0,T]$ as a consequence of the estimates that shall be obtained in the next step.

Estimates. Taking $v = u'_{im}(t)$ in $(2.14)_1$, we have after some calculations that

$$(u_{im}) \text{ is bounded in } L^{\infty}(0,T;V) \tag{2.15}$$

and

$$(u'_{im}) \text{ is bounded in } L^{\infty}(0,T;H), \quad i = 1,...,n. \tag{2.16}$$

To obtain the convergence of the nonlinear term we observe that from Theorem of Lions-Aubin there exists a subsequence (u_{im}) such that:

$$u_{im} \to u_i \text{ strong in } L^2\left(0,T;L^2(\Omega)\right). \tag{2.17}$$

From (2.17) and the continuity of the f follows that

$$f_i(u_{im}(x,t)) \to f_i(u_i(x,t)) \text{ a.e. in } Q. \tag{2.18}$$

We also have, from Remark 1, that $f_i(u_{im}) \in L^2(Q)$, for $i = 1, 2, ..., n$. Therefore, by Lemma of Lions we get

$$f_i(u_{im}) \rightharpoonup f_i(u_i) \text{ in } L^2\left(0,T;L^2(\Omega)\right) \text{ weak as } m \to +\infty, \text{ for } 1 \le i \le n. \tag{2.19}$$

From (2.15), (2.16) and the convergence in (2.19), it is permissible to pass to the limit in $(2.14)_1$ and obtain, by standard procedure, a solution u satisfaying (2.10) − (2.13). ■

Proof of Theorem 3.1. To prove the Theorem 2.1, it is sufficient to prove that u satisfies (2.10) − (2.13) if, and only if, u satisfies (2.6) − (2.9). In fact, we consider the following functional:

$$L(v) = \langle u'' - \Delta u + F(u), v\rangle_{(H^{-1}(\Omega))^n \times \left(H_0^1(\Omega)\right)^n},$$

where u is the solution given by the Lemma 2.1. Observe that $L \in \left(H^{-1}(\Omega)\right)^n$. From (2.12) we have that $L(v) = 0$ for all $v \in V$, in $\mathcal{D}'(0,T)$. By the De Rham's lemma [2], there exists $p \in L^2(\Omega) \diagup \mathbb{R}$ such that

$$L(v) = \int_\Omega p div\ v dx, \quad \forall v \in V.$$

In particular, for all $\varphi \in \mathcal{V} = \{\varphi \in (\mathcal{D}(\Omega))^n;\ div\ \varphi = 0\}$. So, we obtain

$$\langle u'' - \Delta u + F(u), \varphi\rangle = \langle -\nabla p, \varphi\rangle,$$

this is,

$$u'' - \Delta u + F(u) = -\nabla p \quad \text{in} \quad \mathcal{D}'\left(0,T;(\mathcal{D}'(\Omega))^3\right).$$

Reciprocally, if u satisfies (2.6) − (2.9), then multiplying (2.8) by $\varphi \in \mathcal{V}$ and integrating in Ω, we conclude that

$$(u''(t), \varphi) + ((u(t), \varphi)) + (F(u), \varphi) = -\langle \nabla p, \varphi\rangle, \text{ in } \mathcal{D}^{'}(0,T), \ \forall \varphi \in V,$$

thus the theorem is proved. ■

2.1 Energy Conservation

From the weak formulation of the (2.4), taking $v = u'$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left[|u_i'(t)|^2 + \|u_i(t)\|^2\right] + (f_i(u_i(t)), u_i'(t)) = 0. \tag{2.20}$$

If we represent by $G_i(s)$ the function

$$G_i(s) = \int_0^s f_i(r)\,dr$$

then

$$(f_i(u_i(t)), u_i'(t)) = \frac{d}{dt}\int_\Omega G_i(u_i(t))\,dx. \tag{2.21}$$

Substituing (2.21) in (2.20) we obtain

$$\frac{d}{dt}E(t) = 0,$$

where $E(t) = \frac{1}{2}\left[|u_i'(t)|^2 + \|u_i(t)\|^2\right] + \int_\Omega G_i(u_i(t))\,dx$. Therefore $E(t) = E(0) = E_0$.

3 Main Result

In what follows we shall prove (1.3) under the assumption

$$m(x)\cdot\nu \geq \gamma > 0, \quad m(x) = x, \text{ for } x \in \Gamma. \tag{3.22}$$

Theorem 3.1 *We assume that* (1.2) *and* (3.22) *holds true. Then, if* u *denotes the solution of* (1.1) *with initial data* $\{u_0, u_1\} \in V \times H$*, one has*

$$\frac{\partial u}{\partial \nu} \in \left(L^2(\Sigma)\right)^n. \tag{3.23}$$

Proof: In the follows, let us to prove (3.23) for u_0, u_1 smooth, this is, u_0, $u_1 \in \mathcal{U}$ where

$$\mathcal{U} = \{\varphi \in (\mathcal{D}(\Omega))^n \text{ such that } div\ \varphi = 0\}.$$

If we denote by u_μ the solution associated to $u_{0\mu}$, $u_{1\mu} \in \mathcal{U}$ such that,

$$\begin{aligned} u_{0\mu} &\to u_0 \text{ in } V \\ u_{1\mu} &\to u_1 \text{ in } H, \end{aligned}$$

than we have

$$\begin{aligned} u_\mu &\in L^\infty\left(0,T;H^2(\Omega)\cap V\right) \\ u'_\mu &\in L^\infty(0,T;V) \\ u''_\mu &\in L^\infty(0,T;H), \end{aligned} \tag{3.24}$$

and so the integrations by parts to follow are valid.

Multiply both sides of $(2.4)_1$ by $m(x)\cdot\nabla u_\mu = \sum_{k=1}^{n} m_k \dfrac{\partial u_{\mu i}}{\partial x_k}$, and integrate on Q, we obtain:

$$\int_Q \left(u''_{\mu i} m_k \frac{\partial u_{\mu i}}{\partial x_k} - \Delta u_{\mu i} m_k \frac{\partial u_{ki}}{\partial x_k} + f_i(u_{\mu i})\, m_k \frac{\partial u_{\mu i}}{\partial x_k}\right) dxdt = -\int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial u_{\mu i}}{\partial x_k} dxdt \tag{3.25}$$

Remark 3.1 *If we integrate by parts the last term of the* (3.25) *we obtain*

$$-\int_Q \frac{\partial p}{\partial x_i} m_k \frac{\partial u_{\mu i}}{\partial x_k} dxdt = 0.$$

Now we will analyze each term of the first member of the (3.25).

- $\displaystyle\int_Q u''_{\mu i} m_k \frac{\partial u_{\mu i}}{\partial x_k} dxdt = \left(u'_{\mu i}, m_k \frac{\partial u_{\mu i}}{\partial x_k}\right)\Bigg|_0^T + \frac{n}{2}\int_Q \left(u'_{\mu i}\right)^2 dxdt.$

- $\displaystyle -\int_Q \Delta u_{\mu i} m_k \frac{\partial u_{ki}}{\partial x_k} dxdt = -\frac{1}{2}\int_\Sigma m_k \cdot \nu_k \left|\frac{\partial u_{\mu i}}{\partial \nu}\right|^2 d\Sigma + \frac{1-n}{2}\int_Q \left|\frac{\partial u_{\mu i}}{\partial x_k}\right|^2 dxdt.$

Remark 3.2 *For the analysis of the nonlinear term we use the Strauss approximation cf. [10] and [8] to prove that* G_i *is* C^1 *with bounded derivative and* $G_i(0) = 0$*, this is,* $G_i(u_i) \in L^2\left(0,T;H_0^1(\Omega)\right)$*, for* $i = 1, 2, ..., n$*. Therefore we have,*

- $\int_Q f_i(u_{\mu i}) m_k \frac{\partial u_{\mu i}}{\partial x_k} dxdt = -n \int_Q G_i(u_{\mu i}) dxdt.$

Substituing the estimates above in (3.25) and from (3.22) we obtain, after some calculations that

$$\begin{aligned} &\frac{\gamma}{2}\int_\Sigma \left|\frac{\partial u_{\mu i}}{\partial \nu}\right|^2 d\Sigma \leq nTE_0 + 2n\left|\int_Q G_i(u_{\mu i}(x,t))\,dxdt\right| + \\ &+\left(u'_{\mu i}(t), m_k \frac{\partial u_{\mu i}(t)}{\partial x_k}\right)\Bigg|_0^T. \end{aligned} \tag{3.26}$$

From (3.26) follows that

$$\int_\Sigma \left|\frac{\partial u_{\mu i}}{\partial \nu}\right|^2 d\Sigma \leq C(n,T,\Omega)\,E_0, \tag{3.27}$$

because $\left|\int_Q G_i(u_{\mu i}(x,t))\,dxdt\right| \leq C(\Omega)$.

Therefore, from (3.27) we have

$$\frac{\partial u_{\mu i}}{\partial \nu} \longrightarrow \chi \text{ in } L^2\left(0,T;L^2(\Gamma)\right). \tag{3.28}$$

Now observe that by (3.24) and Theorem of Lions-Aubin there exists a susbsequence $(u_{\mu i})$ such that

$$u_{\mu i} \longrightarrow u_i \text{ strong in } L^2\left(0,T;H^1(\Omega)\right). \tag{3.29}$$

By the continuity of the trace we have

$$\int_0^T \|\gamma_1(u_{\mu i}(t)) - \gamma_1(u_i(t))\|^2_{H^{-1/2}(\Gamma)}\,dt \leq C\int_0^T \|u_{\mu i}(t) - u_i(t)\|^2_{H^1(\Omega)}\,dt, \tag{3.30}$$

for each $i = 1, ..., n$.

From (3.29) and (3.30) we conclude that

$$\frac{\partial u_{\mu i}}{\partial \nu} \longrightarrow \frac{\partial u_i}{\partial \nu} \text{ in } L^2\left(0,T;H^{-1/2}(\Gamma)\right) \text{ as } \mu \to +\infty,\ i = 1, ..., n. \tag{3.31}$$

From (3.28) and (3.31) follows that

$$\chi = \frac{\partial u_i}{\partial \nu} \in L^2(\Sigma)\,,\ \text{for each } i = 1, ..., n,$$

this is,

$$\frac{\partial u}{\partial \nu} \in \left(L^2(\Sigma)\right)^n.$$

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