Arboreal Extension of an Order

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Abstract. *This article describes the contribution of Adriana Pimenta, Jayme Szwarcfiter, Michel Habib and Sulamita Klein on arboreal extensions of partial orders, started in 2008. The quantity to be determined is about the minimum number of arboreal jumps of a partial order.*

Resumo. *Este artigo descreve a contribuição de Adriana Pimenta, Jayme Szwarcfiter, Michel Habib e Sulamita Klein sobre extensões arbóreas de ordens parciais, iniciado em 2008. O problema a ser determinado é sobre o número mínimo de saltos de extensões arbóreas de uma ordem parcial.*

1. General Information

In mathematics, especially order theory, a partially ordered set, or Poset, formalizes and generalizes the intuitive concept of an ordering of the elements of a set. A poset consists of a set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. An ordered pair $P = (X, \leq_P)$ is said to be a partially ordered set when there is a binary relation $\leq_P$ between the elements of X that is reflexive, antisymmetric and transitive, that is, for all $x, y, z \in X$, we have:

1. Reflexivity: $x \leq_P x$;
2. Antisymmetry: if $x \leq_P y$ and $y \leq_P x$ then $x = y$;
3. Transitivity: if $x \leq_P y$ and $y \leq_P z$ then $x \leq_P z$.

A *chain* is a totally ordered subset of a poset; an *antichain* is a totally unordered subset. We say that y *cover* x, $x, y \in X$, if $x < y$ and $x \leq r < y$ implies $x = r$. Let us denote $C^+(y) = \{x \in X; y \text{ is covered by } x \text{ in } P\}$; $C^-(y) = \{x \in X; y \text{ cover } x \text{ in } P\}$. We define the ideal of y by $I(y) = \{x \in X; x < y\}$ and the *filter* of y by $F(y) = \{x \in X; y < x\}$, where $y \in X$.

We say that $A = (X, A)$ is an *arboreal order* if for all $y \in X$; $I(y)$ is a chain.

2. The problem

Given any partial orders $P = (X, \leq_P)$ and $P' = (X, \leq_{P'})$ on a set X, we say that P' is an *extension* of an order P exactly when $P \subset P'$, that is, for all $x, y \in X$ if $x \leq_P y$ then $x \leq_{P'} y$. Alternatively, an extension P' may be viewed as an order preserving bijection

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from a partially ordered set P on the same set X. In particular, if P' is a chain then it is called a linear extension of P.

Let be A an extension of an order P, a pair (p_i, p_{i+1}) forms a *jump* in A if p_i is not covered by p_{i+1} in P or equivalentely, a *jump* of A is a relation contained in the Hasse diagram of A, but not in the order P.

An extension A of an order P is arboreal if its Hasse diagram is a rooted tree with a unique minimal element. The *arboreal jump number* of A is the number of jumps contained in it, we denote by $s_A(P)$. We study the problem of finding the arboreal extension of P having minimum arboreal jump number—a problem related to the well-known (linear) jump number problem.

We study the problem of finding the arboreal extension A of an order P that has a minimum number of new relations added to the Hasse diagram of P. Moreover, we state the problem of finding the arboreal extension that has the minimum number of arboreal jumps.

3. Complexity

We examine the complexity of the arboreal jump number problem. The input of this problem consists of an order P and an integer k. The question is whether P admits an arboreal extension A having at most k jumps. The linear jump number problem is similar, except that the aim is to find a linear extension, instead.

Given an order P and an integer k, it is $NP-$complete to decide if $s_A(P) \leq k$, even if P is an interval order, where $s_A(P)$ is its arboreal jump number.

4. Characterization of Minimal Arboreal Extensions

Arboreal orders can be characterized in terms of forbidding a simple structure. Let $P(X, P)$ be an order and $v, x_1, x_2 \in X$. Say that v, x_1, x_2 form a violation of P, when $x_1, x_2 \in C^-(v)$. In this case, name v as a violator. We observe that, an order is arboreal if and only if it does not contain violations.

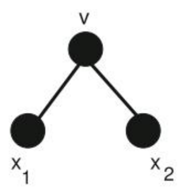

Figure 1. Forbidden structure of an arboreal order: violator

Next, we examine minimal arboreal extensions. First, we describe a characterization of such extensions. Let $P(X, P)$ be an order and A an arboreal extension of it. Then A is minimal if and only if for every jump (y, z) of A there are elements $v, x_1, x_2 \in X$ such that

$$y \in F_A[x_2] \backslash F_A[x_1],$$

$$z \in I_A[x_1] \backslash I_A[x_2]$$

and

$$x_1, x_2 \in C_A^-(v),$$

where $A = A - (y, z)$.

5. Conclusions

In [3] we have considered the problem of determining the minimum number of jumps of an arboreal extension A of a given order P. Different results concerning this problem have been described, including NP-completeness, polynomial time special cases, bounds and the concept of minimal extensions. In the problem considered in the present paper, a jump has been defined as a relation belonging to the Hasse diagram of an arboreal extension A of P, but not belonging to the order P

This work was developed between 2008 and 2010. Based on the studies, we present the first results at the XLI SBPO that took place in Porto Seguro - BA, in 2009 [1]. In 2012 we present more results at the LAWCG that took place in Buenos Aires - Argentina [2]. In 2013, the results obtained in collaboration with researchers Jayme Szwarcfite, Sulamita Klein, Michel Habib and Adriana Pimenta were published in Volume 30, Number 1 - A Journal on the Theory of Ordered Sets and their Applications - ORDER by Springer.

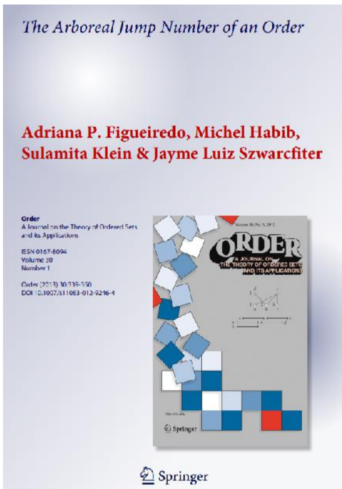

Figure 2. The-Arboreal-Jump-Number-of-an-Order.pdf

6. Thanks

Adriana: My special thanks for my academic achievements goes to Professor Jayme Szwarcfiter who actively participated, collaborating and teaching me to be the professional I am today and to everyone who participated in this process. Finally, thank you all for the opportunity to commemorate Professor Jayme's life.
Sula: My special thanks to Jayme for the opportunity to co-advise Adriana. It is always a pleasure and an apprenticeship to work with Jayme.

Figure 3. Jayme, Adriana and Sula.

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