Interval Count of Interval Graphs

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Abstract. *The interval count problem is that of determining the smallest number of distinct interval lengths that is sufficient to represent an interval model of a given interval graph. The class of interval graphs is well known, with several applications. This article briefly summarizes my doctorate thesis done on the subject with supervision of Prof. Jayme Szwarcfiter, in his honor, on the occasion of his 80th birthday.*

1. A long standing problem

Interval count is a long stand problem – at least, for me. It is was one of my first research problems that Jayme suggested me to work on when I was starting my graduation course. After almost one decade now working with him on this problem, I can no longer say which I like more about it: the pleasure of studying this difficult problem, or the joy of collaborating with Jayme on it.

An *interval model* is a set of closed intervals on the real line. An interval model $\mathcal{M}$ is said to be *of* some graph G if it is possible to assign each $v \in V(G)$ to a distinct interval $I_v \in \mathcal{M}$ such that $(u, v) \in E(G) \iff I_u \cap I_v \neq \emptyset$; in this case, we call G an *interval graph*.

Naturally, an interval model of a given interval graph G is not unique (there are actually infinitely many). Nevertheless, on facing the problem of picking one interval model of an interval graph, some interval models may be of more interest than others due to some practical (or theoretical) application. Among all optimization criteria to select the "best" interval model, one comes quite naturally: choosing one that minimizes the number of distinct interval lengths. Formally, if $\text{IC}(\mathcal{M})$ denotes the number of distinct interval lengths in the interval model $\mathcal{M}$, the *interval count* of the interval graph G is

$$\text{IC}(G) = \min\{\text{IC}(\mathcal{M}) \mid \mathcal{M} \text{ is an interval model of } G\}$$

This "natural" optimization problem actually was firstly suggested by the famous mathematician Ronald Graham in the 1980s. Figure 1 presents an example of a graph G and two interval models of G with different numbers interval lengths. The interval model in Figure 1(ii) has three different lengths and the one in Figure 1(iii) has two different lengths. It is possible to prove that there is no interval model of G with only one distinct length, and thus the interval count of G is equal to 2, that is, $\text{IC}(G) = 2$.

Determining whether $\text{IC}(G) = 1$ by an efficient deterministic algorithm is a well known problem, equivalent to determining whether G is a unit interval graph, which has been solved since the 70s [Roberts. 1969, Cornell et al. 1995, Bogart and West 1999,

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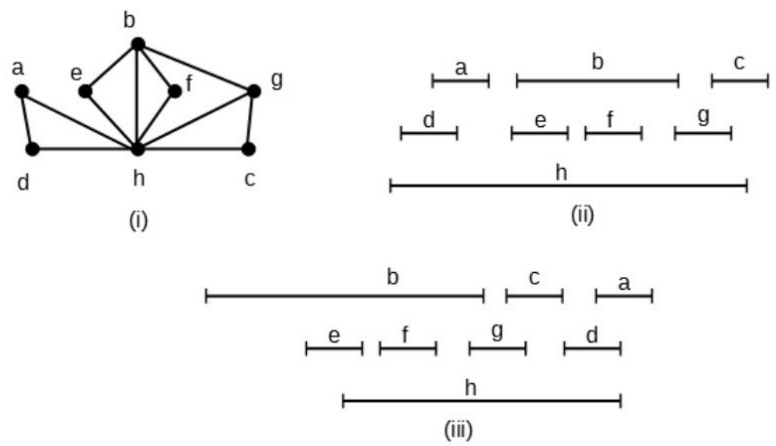

Figure 1. (i) an interval graph and (ii-iii) some of its interval models.

Gardi 2007]. However, determining whether IC$(G) = k$ for any fixed $k \geq 2$ by an efficient deterministic algorithm is open [Cerioli et al. 2012, Klavík et al. 2019].

2. Doctorate thesis results

Although my doctorate thesis approached other graph problems, an important part of it was devoted to the interval count problem [Oliveira 2011].

We showed that interval models can be assumed to have distinct integer endpoints without affecting the interval count of the corresponding graph. Such an assumption is, in general, quite natural to problems related to interval graphs, but it is not straightforward to be assumed for interval counting without a careful thought. Indeed, slightly moving the interval endpoints, in an attempt to transform a given interval model with non-integer or non-distinct endpoints into another with distinct integer endpoints, potentially changes the number of distinct interval lengths. The aim of this proof is then describe a safe procedure to do so. This leads to a potential simplification in the solution of other problems involving interval count.

We also discussed the combinatorial aspect of this problem. Brute force algorithms with exponential time complexities are generally evident for combinatorial problems. However, even using the previous result, it is still not clear how to use a brute force algorithm to solve the interval count problem. Even under the assumption that the endpoints of the intervals are integers, as there is no known upper limit for the total length of such interval models, one can not test all possible interval models having integer endpoints. We proposed a decomposition of the problem into a finite, although exponential, series of linear programming problems. These formulations were used later on [Joos et al. 2014] to provide a polynomial algorithm to solve the problem of when a bipartition $X \cup Y$ of the vertex set of an interval graph G is given, and one asks whether there exists an interval model of G in which all vertices of X (resp. Y) have the same length.

Finally, we worked on the interval count problem when the graph is constrained to belong to certain classes. In the literature, it was known at that time that trees, almost claw-free graphs, and threshold graphs have all interval count 2. We extended this result to the class of generalized threshold graphs. Furthermore, we showed that the classes of trivially perfect and bull-extended free graphs have examples of graphs with arbitrary interval count values. Most importantly, we provided efficient algorithms for determining the interval count of graphs in such classes. A byproduct of this result was that we provided an efficient algorithm for determining the nesting depth of an interval graph, which is the minimum k for which there exists an interval model of such a graph having a chain of at most k nested intervals [Cerioli et al. 2011]. (In an interval model $\mathcal{M}$, an interval $I_a \in \mathcal{M}$ is said to be nested in an interval $I_b \in \mathcal{M}$ if there are other two intervals $I_x, I_y \in \mathcal{M}$ such that I_b intersects I_a, I_x, I_y and I_a lies entirely between I_x and I_y.) This result has recently been improved to linear time [Klavík et al. 2019].

3. Ongoing work

We have been working on different results on interval count since my thesis. I have now the honor to supervise along with Jayme a doctorate student on the theme. As a result of this thesis, we have looked at different other aspects of interval count [Francis et al. 2022, Medeiros et al. 2019, Medeiros et al. 2021b, Medeiros et al. 2021a].

This brief article is written in honor of Jayme Szwarcfiter, on the occasion of his 80th birthday. I would like to sincerely thank him for all the mentoring, collaboration, and friendship over all these years.

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