Chordal graphs through clique trees

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Abstract. *Here we present a part of the work in collaboration with Jayme Luiz Szwarcfiter and Marisa Gutierrez, during the development of the thesis of Silvia Tondato at Facultad de Ciencias Exactas (UNLP), "Grafos Cordales: árboles cliques y representaciones canónicas", finished in 2009.*

1. Introduction

Chordal graphs form an important and well studied class of graphs. It is known that for each chordal graph there is a tree T, called clique tree, whose vertices is the set $\mathcal{C}$ of all maximal cliques of a graph, and for every vertex v in the graph, the set $\mathcal{C}_v$ of maximal cliques that contain it induces a subtree of the clique tree, denoted by $T(\mathcal{C}_v)$. It is interesting from theoretical point of view and potentially beneficial from a practical standpoint to consider how a clique tree representation may be better than another in a given context.

Clique trees have many applications (Zotenko, Guimarães, Jothi and Przytycka), for example it can be used to study protein interactions. Most cellular processes are carried out by multi-proteins complexes, groups of proteins that bind together to perform a specific task. A better understanding of this organization of proteins into overlapping complexes is an important step to unveiling functional and evolutionary mechanisms behind biological networks. This situation can be represented by a graph where the vertices are proteins and two vertices are connected by an edge if the corresponding proteins interact. Complex proteins can be seen as cliques of this graph. Then, when the graph is chordal, a clique tree and the family of subtrees representing the vertices provide a good framework for following the activity of a protein in different complexes.

Several subclasses of chordal graphs have been characterized in terms of their clique trees (Monma and Wei). Thus an **UV** graph G is a chordal graph which has a clique tree T satisfying that for every vertex v of G, $T(C_v)$ is a path in T, a **DV** graph G is a chordal graph which has a directed clique tree T satisfying that for every vertex v of G, $T(C_v)$ is a directed path in T, and an **RDV** graph G is a chordal graph which has a rooted directed clique tree T satisfying that for every vertex v of G, $T(C_v)$ is a directed path in T.

A structural problem about these subclasses that has been of interest is to obtain characterizations by forbidden subgraphs. For example, there are characterizations of this type for the following classes of chordal graphs: interval graphs (Lekkerkerker and Boland), proper interval graphs (Roberts), strongly chordal graphs (Farber), split graphs (Földes and Hammer), threshold graphs (Chvátal and Hammer), directed path graphs (Panda), starlike graphs and starlike-threshold graphs (Cerioli and Szwarcfiter)

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The study of chordal graphs or subclasses of chordal graphs is equivalent to analyze clique trees. In the thesis of Silvia Tondato at Facultad de Ciencias Exactas UNLP, the relationship between the clique trees of a chordal graph, its subgraphs and some subclasses are considered.

2. Study of asteroidal set of chordal graphs by properties of clique trees

In [Gutierrez et al. 2003], [Gutierrez et al. 2007] and [Gutierrez et al. 2008a], we studied the relationship between the clique trees of a chordal graph and its subgraphs. As a consequence of this study, clique trees can be described locally using two technical tools that consist of cutting and pasting clique trees. Thus all clique trees of a graph can be obtained from clique trees of subgraphs. In particular we were interested in the leafage of a chordal graph, which is the minimum number of leaves among all the clique trees of the graph. It is clear that connected interval graphs are exactly those chordal graphs with leafage 2. In a historical work Lekkerkerker and Boland proved that interval graphs are the chordal graphs without 3-asteroidal (3 vertices such that between any two of them there is a path avoiding the neighborhood of the third). We proved a generalization of this result using two tools: cutting and pasting clique trees. Our result is that in a clique tree that realizes the leafage, for every vertex of degree at least 3, and every choice of 3 branches incident to it, there is a 3-asteroidal in these branches.

3. Forbidden subgraph characterization

A class **C** of graphs is said to admit a forbidden subgraph characterization if there exists a class $\mathcal{F}_{\mathbf{C}} = \{H : H$ is not in **C** but H -v is in **C** for every v in V(H)$\}$ such that $G \in$ **C** iff G does not contain any member of $\mathcal{F}_{\mathbf{C}}$ as an induced subgraph. If **C** admits forbidden subgraph characterization, then $\mathcal{F}_{\mathbf{C}}$ is called the class of minimal forbidden subgraphs. It is well known that a class **C** admits forbidden subgraph characterization iff **C** is closed under vertex-induced subgraphs. For many classes of graphs, it follows from the definition that the class is closed under vertex-induced subgraphs; in such cases, there must be a forbidden subgraph characterization. However, finding the explicit forbidden subgraph characterization can be difficult.

In [Gutierrez et al. 2005] and [Gutierrez et al. 2008b], we studied **UV** graphs and **RDV** graphs. In particular, for **UV** graph we searched conditions in clique tree in order to avoid claws. These conditions leaded to the description of forbidden subgraphs.

With respect to **RDV** graphs, we proved that there is not any minimal induced subgraph with false twins (false twins are vertices which are not adjacent but have the same neighbors).

4. Conclusions

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