When Biclique graphs were born

Marina Groshaus[1,2]

[1]DAINF, Universidade Tecnológica Federal do Paraná (UTFPR)
Curitiba PR, Brazil

[2]CONICET – Argentina

marinagroshaus@yahoo.es

Abstract. *This paper is a survey of my work about biclique graph and biclique properties developed during my PhD with Jayme Szwarcfiter and some related results.*

1. Introduction

So this is the 80'th birthday of Jayme!! How to start talking about him? Going back in time, it takes me to his 70'th birthday which was celebrated in the LAWCG´2022 of which I was the chair.

It is a picture in my memory the wall covered by more than a hundred of the first pages of Jayme's.

It was in that celebration that I met my husband, professor André Guedes, and, as a magic coincidence, chair of the 80´th birthday – LAWCG 2022. I will be always grateful to Jayme for that. Going a bit more back in time, I arrive at Copacabana beach, were I see me walking and making drawings of graphs in the sand looking for an algorithm to solve the problem Jayme had suggested me in the day before. Jayme was my PhD adviser and I used to travel twice a year to meet him. He opened his house for me, inviting me for family dinners and even going together to take his daughter to a party, so we could talk about work on a Saturday afternoon!

Our first contact was in a visit to Rio in 2001 and where he proposed to study the biclique graph operator for my PhD. Bicliques and I was love at first sight, and although the proposal was very challenging, in particular because of the distance Rio-Buenos Aires, I accepted. As advisor, he gave me a lot of freedom, accepting every subjects I proposed to study. It is worth mentioning that our work went out this way: I used to send theorems and proofs and he used to answer saying that they were very good results but they needed some writing corrections. Then, he used to send my pdf with many many many corrections, and most of the times, he used to re-wrote the proofs entirely. I gave him so much work!!!!! I can tell the dedication of his reading to every word I put down on my thesis and the dedication to myself, worrying about every detail of my days in Rio. He behave as a father and I will always be grateful for that.

2. Bicliques, biclique graph brat boy and its friends

Bicliques in graphs are defined as maximal bipartite complete induced subgraphs. The biclique graph $KB(G)$ appears as a cousin of the clique graph, defined as the intersection graph of bicliques of G. Our challenge was to characterize graphs that are bicliques

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graphs. For that, we first characterize biclique matrices and with that, we gave a characterization of bicliques graphs [Groshaus and Szwarcfiter 2010]. This characterizations is similar to that for clique graphs given in [Roberts and Spencer 1971], but using a different approach on the proof. Sadly, neither of those characterizations lead a polynomial time algorithm. While in [Alcón et al. 2006] is proved that the clique graph recognition problem is NP-complete, it still is an open problem to discover the computational complexity of recognizing biclique graphs.

During my PhD journey, we defined the bipartite-Helly property similar to the Helly property but focus on bipartite families and gave a polynomial recognition algorithm for bipartite Helly labelled sets [Groshaus and Szwarcfiter 2011].

Since bicliques were, at that point, part of my dreams (literally), we got excited with the idea of considering other problems that already existed related to cliques, but in the context of bicliques. That is how the idea of studying the Helly property in the family of bicliques of a graph appeared. I already knew that leading with bicliques would be much more complicated that its closer cousin, the cliques. Nevertheless, we went forward and considered the biclique Helly graphs. In [Groshaus and Szwarcfiter 2007] we gave a polynomial algorithm for the problem of recognizing biclique Helly graphs following the ideas of the characterization of clique Helly graph given by Jayme in [Szwarcfiter 1997].

Then, in [Groshaus and Szwarcfiter 2008] we focused on the Helly property but in an hereditary context and we gave a polynomial algorithm to recognize biclique Helly hereditary graphs along with a characterization by forbidden subgraphs.

Jayme opened the door for the a deeper study of bicliques in many contexts, which were not explored at the time. I followed the way that my academic father dragged and went on: We studied the biclique operator behaviour along with my student Leandro Montero and professor Andre Guedes [Groshaus et al. 2016].

Immediately we realized that the biclique operator was a real rebel one! Its behaviour was very messy since for most of the graphs, it adds many vertices and edges.

Knowing that the behaviour of the clique graph operator was not well solved, we were skeptic about what could happened with this uncontrollable operator. We discovered that thanks to its messy behaviour, we could answer in linear time if a graph diverges or not by the biclique operator. Moreover, we proved that: “almost every graph diverges!”, were “almost” means that only a finite family of graphs are convergent and no graph cycles under the biclique operator (considering twin-free graphs). This results along with the characterization of graphs that diverge appears in [Groshaus and Montero 2013, Groshaus et al. 2016].

In the last years, some improvements about properties and the recognition problem of biclique graphs were made, considering subclasses as triangle free graphs and split graphs [Groshaus and Montero 2021], [Cruz et al. 2020, Groshaus et al. 2022].

The last news about the biclique operator is resumed in [Groshaus and Guedes 2020] where Table 1 can be found. It summarizes what it is know so far about recognition of biclique graphs of some graph classes.

A long way, started by the seed of Jayme´s idea, is still to be walked.

class $\mathcal{A}$	$KB(G)$**,** $G \in \mathcal{A}$	**class** $KB(\mathcal{A})$	**complexity**
complete	$L(G)$	L(complete)	$\mathcal{P}$
tree	$(G - leaves(G))^2$	$(\text{tree})^2$	$\mathcal{P}$ (linear)
path (P_n)	$\emptyset$, for $n = 1$ K_1, for $n = 2$ $(P_{n-2})^2$, for $n > 2$	$(\text{path})^2$	$\mathcal{P}$ (linear)
caterpillar (tree)	$(G - leaves(G))^2$	$(\text{path})^2$	$\mathcal{P}$ (linear)
cycle (C_n)	K_1, for $n = 4$ $(C_n)^2$, for $n \neq 4$	$(\text{cycle})^2 - K_4 + K_1$	$\mathcal{P}$
$\mathcal{G}_k$, for $k \geq 5$ (*)	$(G - leaves(G))^2$	$(\mathcal{G}_k)^2$, for $k \geq 5$	$\mathcal{P}$, for $k \geq 6$ $\mathcal{NP}$**-complete**, for $k = 5$
IBG	**OPEN**	$\subset$ (IIC-PG)^2 $\subset$ $K_{1,4}$-free co-CG	**OPEN**
PIB	$(L(S(G))^2$	$(L(PIB))^2$	**OPEN**
PIB-ASG	$(L(S(G))^2$	1-PIG	$\mathcal{P}$
HIB	$K(G^2)$	$\subset PIG \cap (L(PIB))^2$	**OPEN**
$\{K_3, C_5, C_6\}$-free	$K(G^2)$	**OPEN**	**OPEN**
$BBHGD$	**OPEN**	$CHBDI$	**OPEN**
$NSSG$	**OPEN**	**OPEN**	$\mathcal{P}$
threshold	**OPEN**	**OPEN**	$\mathcal{P}$
K_3-free	$(KB_m(G))^2$	$\subset \mathcal{G}^2$	**OPEN**
bipartite	$(KB_m(G))^2$	$(IIC\text{-comparability})^2$	**OPEN**
$\mathcal{G}$	**OPEN**	Characterization	**OPEN**

Table 1. At column "$KB(G)$, $G \in \mathcal{A}$" a brief description of $KB(G)$; at column "class $KB(\mathcal{A})$", class that is equal to (or a super-class of) $KB(\mathcal{A})$; at column "complexity", complexity (if known) of recognizing $KB(\mathcal{A})$.

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