Why are they called Trivially Perfect Graphs?

Martin Charles Golumbic[1]

[1] University of Haifa, Mt. Carmel, Haifa, Israel

golumbic@cs.haifa.ac.il

Dedicated to Jayme Luiz Szwarcfiter on the occasion of his 80th birthday

Abstract. *Trivially perfect graphs are graphs for which the stability number equals the number of (maximal) cliques, for every induced subgraph. Trivially perfect graphs are equivalent to the $\{C_4, P_4\}$-free graphs, and to the comparability graphs orders whose Hasse diagram is a rooted tree. We survey classical and recent results on this fascinating and highly non-trivial graph family.*

1. Dagstuhl, February 9, 2014

Trivial perfect graphs are characterized as being the $\{C_4, P_4\}$-free graphs—so stated the young speaker at the opening lecture of the Dagstuhl workshop *Graph Modification Problems*. A hand went up, "Why are they called trivial perfect graphs?" A smile came to my face. I was glad someone asked. I said nothing, but would answer two days later.

2. Paris 1977

Let $mc(G)$ denote the number of maximal cliques of an undirected graph G, and let $\alpha(G)$ be the stability number, that is, the cardinality of the largest set of independent set. Clearly,

$$\alpha(G) \leq mc(G) \qquad (1)$$

since there must be $\alpha(G)$ distinct cliques, each one containing a different member of a maximum stable set. [Buneman 1974] had falsely stated that equality holds in (1) for chordal graphs. In fact, equality is not even true for a chordless path $P_n (n \geq 4)$, which lead me to the question, "For which graphs is there equality in (1)?" Unfortunately, one cannot expect to discover much about the structure of such graphs, since we can hide all the structure of G as follows:

Let $C_1, C_2, \ldots, C_m$ be the maximal cliques of G, and add m new vertices $x_1, x_2, \ldots, x_m$ connecting x_i to each vertex of C_i to form an augmented graph H. Clearly, $\alpha(H) = mc(H) = m$ thus satisfying (1) with equality, yet any structure in G is totally lost. For this reason, in my paper [Golumbic 1978], I added a hereditary condition to define something structurally interesting.

Definition: A graph $G = (V, E)$ is *trivially perfect* if $\alpha(G_X) = mc(G_X)$ for all induced subgraphs G_X of G, $(X \subseteq V)$.

Here is a quote from that paper. "This name was chosen since it is trivial to show that such a graph is perfect."

Definition: A graph $G = (V, E)$ is *perfect* if $\alpha(G_X) = \kappa(G_X)$ for all induced subgraphs G_X of G, $(X \subseteq V)$, where $\kappa(G)$ denotes the size of a minimum clique cover of G.

Cadernos do IME - Série Informática
e-ISSN: 2317-2193 (online)
DOI: 10.12957/cadinf.2022.70585

It is always the case that a maximum stable set number is smaller than or equal to the size of a minimum clique cover, and trivially, $\kappa(G)$ is at most $mc(G)$. In other words, $\alpha(G_X) \leq \kappa(G_X) \leq mc(G_X)$, so if the first term equals the last, then they are all equal, and G is perfect.

3. Structural characterizations of trivially perfect graphs

The structural characterizations that connect trivially perfect graphs with the rest of the graph theory world are repeated here.

Theorem 1. Let $G = (V, E)$ be an undirected graph. The following statements are equivalent:

(i) G is trivially perfect;

(ii) G contains no induced subgraph isomorphic to C_4 or P_4;

(iii) G has a transitive orientation whose Hasse diagram is a forest of rooted trees;

(iv) G can be formed by repeatedly (1) adding a new isolated vertex, (2) adding a new vertex that is adjacent to all old vertices, or (3) taking the disjoint union of two such graphs;

(v) if $a, b, c, d \in V$ are distinct vertices satisfying $ab, bc, cd \in E$, then either $ac \in E$ or $bd \in E$;

(vi) each connected induced subgraph of G has a universal vertex.

A proof of (i) $\Leftrightarrow$ (ii) can be found in [Golumbic 1978], and it is a simple exercise to show (ii) $\Leftrightarrow$ (v). [Wolk 1965] proved that a connected graph satisfying (v), which he called a *D-graph*, has at least one universal vertex—that is, (v) $\Rightarrow$ (vi). He then used this to show by induction that (vi) $\Rightarrow$ (iii). A simple proof of the implication (iii) $\Rightarrow$ (v) is in [Wolk 1962]. Finally, it is straightforward to show the equivalence (iv) $\Leftrightarrow$ (vi). [Ma, Wallis, and Wu 1989] used the term *quasi-threshold graphs* for graphs satisfying (iv) since G is a threshold graph (no induced C_4 or P_4 or $2K_2$) precisely when G and its complement $\overline{G}$ are trivial perfect. In a recent paper, [Rubio-Montiel 2015] characterizes the trivially perfect graphs in terms of vertex-coloring.

It should be pointed out that the property of having a transitive orientation whose Hasse diagram is a tree is not a comparability invariant. An example of this is the following graph on 5 vertices and 8 edges: (1) one TRO is $ab, ac, ad, ae, bc, bd, be, de$ whose Hasse diagram rooted at a is ab, bc, bd, de, but (2) another TRO is $ab, ac, ad, ae, cb, db, de, eb$ whose Hasse diagram contains a cycle, namely, ac, ad, cb, de, eb.

[Wolk 1965] also observed that the complement of a trivially perfect graph is a comparability graph. This implies that they are a subclass of both permutation graphs and interval graphs. Armed with the characterizations in Theorem 1, it is easy to see that we have the mini-hierarchy in Figure 1. See [Golumbic 2021] for further definitions and references.

4. Algorithmic aspects of trivially perfect graphs

Recognition. [Ma, Wallis, and Wu 1989] studied algorithmic aspects using characterizations of Theorem 1 giving an $O(|V||E|)$-time algorithm for the recognition problem. Subsequently, [Yan, Chen, and Chang 1996] improved the complexity of recognition by

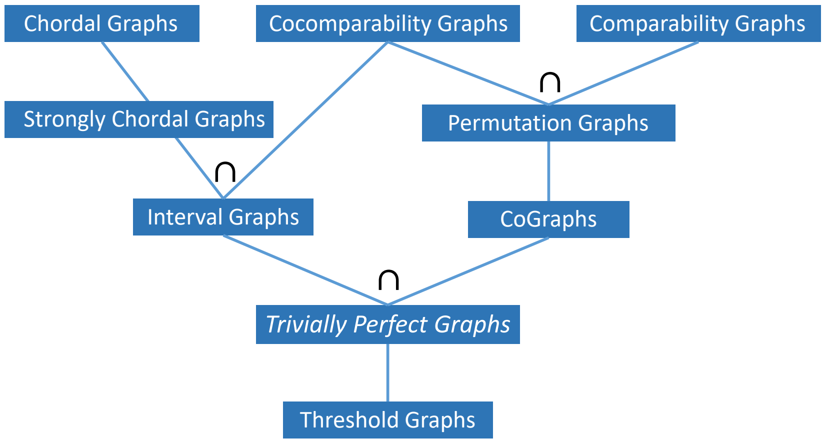

Figure 1. A hierarchy of graph classes

demonstrating a linear-time algorithm. [Chu 2008] then presented a simple linear time algorithm for recognizing trivially perfect graphs, based on lexicographic breadth-first search. Whenever the LexBFS algorithm removes a vertex v from the first set on its queue, the algorithm checks that all remaining neighbors of v belong to the same set; if not, one of the forbidden induced subgraphs can be constructed from v. If this check succeeds for every v, then the graph is trivially perfect. The algorithm can also be modified to test whether a graph is the complement graph of a trivially perfect graph, in linear time. See [Golumbic 2004, 2021] for references to other applications of LexBFS.

Graph sandwich. For a graph class $\mathcal{C}$, the *graph sandwich problem* (GSP) asks whether, given a set of vertices V, a set of mandatory edges E^1, and a set of optional edges E^0, is there a graph $H = (V, E)$ in $\mathcal{C}$ such that $E^1 \subseteq E \subseteq E^1 \cup E^0$? [Alvarado, Dantas, Rautenbach 2019] have recently shown that the trivially perfect graph sandwich problem can be solved in polynomial-time. For the other graph classes in Figure 1, in a series of papers in the 1990s, Golumbic, Kaplan and Shamir proved that the GSP is NP-complete for chordal, strongly chordal, permutation, comparability, and cocomparability graphs, but is polynomial for cographs and threshold graphs.

Square root. A graph H is a *square root* of a graph G if two vertices are adjacent in G if and only if they are at distance one or two in H. This is usually denoted $H^2 = G$. On the one hand, computing a square root of a given graph G is NP-hard, even when the input graph G is restricted to be chordal. On the other hand, [Milanič and Schaudt 2013] have shown that computing a square root can be done in linear time for the class of trivially perfect graphs. They further investigated the problem of determining whether there exists a split graph H such that $H^2 = G$. They prove this split square root problem can be solved in linear time even for the case where G is chordal, and hence for trivially perfect graphs too.

5. Jayme Szwarcfiter

Several papers authored by Jayme Szwarcfiter and colleagues refer to trivially perfect graphs. In [Dobson, Gutierrez, and Szwarcfiter 2004, 2006], the authors study *treelike* *comparability graphs*, that is, comparability graphs of posets whose Hasse diagram is a tree (not necessarily a rooted tree). They give necessary and sufficient conditions that a prime comparability graph must satisfy for being a treelike comparability graph. They also provide a characterization of treelike comparability graphs based on modular decomposition.

Other early works on comparability graphs whose Hasse diagram is a tree are [Arditti 1975] and [Atkinson 1990]. [Cornelsen and Di Stefano 2004, 2009] give another characterization of treelike comparability graphs as being distance hereditary with a special treelike orientation of its split decomposition, and provide a linear-time recognition algorithm. They also characterize treelike permutation graphs.

Jayme Luiz Szwarcfiter delivered the 2nd Annual Uri Natan Peled Memorial Lecture at the Haifa Workshop on Interdisciplinary Applications of Graph Theory, Combinatorics and Algorithms in May 2011. He spoke on the topic, "Arboricity, h-index and dynamic graph algorithms" [Lin, Soulignac, Szwarcfiter 2012]. In this paper, they proposed a data structure for manipulating graphs, called h-graph, which is particularly suited for designing dynamic algorithms. Using this data structure, they solve problems such as listing the cliques of a given size, recognizing diamond-free graphs, and finding simple, simplicial and dominated vertices. They also design new static algorithms for recognizing strongly chordal graphs, an important superclass of trivially perfect graphs.

The clique graph $\mathcal{K}(G)$ of a graph G is the intersection graph of the maximal cliques of G. Given integers $m_1, \ldots, m_k$, the *weighted clique graph* of G is the clique graph in which there is a weight assigned to each complete set X of size m_i of $\mathcal{K}(G)$, for each $i = 1, \ldots, k$. This weight equals the cardinality of the intersection of the cliques of G corresponding to X.

In [Bonomo and Szwarcfiter 2014], the authors characterize weighted clique graphs in a manner similar to that of Roberts and Spencer's for clique graphs. They further characterize several classical graph classes in terms of their weighted clique graphs, providing a common framework for describing some different well-known classes of graphs. These include hereditary clique-Helly graphs, split graphs, chordal graphs, interval graphs, proper interval graphs, trivially perfect graphs, line graphs, among others.

6. Further generalizations of trivially perfect graphs

Motivated by Wolk's characterization of trivially perfect graphs in Theorem 1(v), [Jung 1978] defined a more general family as follows:

Definition: A graph G is called a D^*-*graph* if whenever $a, b, c, d \in V$ are distinct vertices satisfying $ab, bc, cd \in E$, then either $ac \in E$ or $bd \in E$ or $ad \in E$.

In his paper, Jung proved that the larger class of D^*-graphs is also a class of comparability graphs and the involved posets are called multitrees, whose elementary properties are also studied.

[Johnson and McMorris 1982] strengthen the condition of Theorem 1(v) by defining a graph G to be a strong D-graph if whenever $a, b, c, d \in V$ are distinct vertices satisfying $ab, bc, cd \in E$, then both $ac \in E$ and $bd \in E$. They show that G is a strong D-graph if and only if it is the union of fans and chains. Similarly, they define and characterize a notion of strong D^*-graphs.

Further recommended reading is [Tsujie 2018] on the relation between trivially perfect graphs and the chromatic symmetric function of a graph [Stanley 1995, 1998], and [Gurski, Komander, and Rehs 2021] on directed trivially perfect graphs.

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