Generating interval and semi-order extensions of partially ordered sets

Celina Figueiredo[1], **Aquiles Braga de Queiroz**[2]

[1]COPPE – Universidade Federal do Rio de Janeiro (UFRJ)
Caixa Postal 68511 – 21.941-972 – Rio de Janeiro – RJ – Brazil

[2]IM-DTL – Universidade Federal Rural do Rio de Janeiro (UFRRJ)
Rio de Janeiro – RJ – Brazil

`celina@cos.ufrj.br, abq@ufrrj.br`

Abstract. *This is an account of the collaboration and the research conduction by Prof. Jayme Luiz Szwarcfiter, during the development of the thesis of Aquiles Braga de Queiroz at COPPE/UFRJ, "Algoritmos para a Geração de Classes de Extensões de Conjuntos Parcialmente Ordenados", completed in 2005.*

1. Introduction

Characterizations of extensions of a partial order P on a set E — through bijections relating them to substructures obtained from P — have been presented by several authors. Bonnet and Pouzet establish a bijection between linear extensions of a partial ordering and maximal chains of its ideal lattice [Bonnet and Pouzet 1969]. Additionally, Bonnet and Pouzet present a characterization of the set of extensions of P defined by forbidden convex structures. Habib, Morvan, Pouzet and Rampon [Habib et al. 1991] present a bijection between maximal chains in the lattice of maximal antichains of P and minimal interval extensions of P. Baldy, Morvan and Thierry [Baldy et al. 1999] present a one-to-one correspondence between the set of all extensions of a partial order P and the set of all sublattices of the ideal lattice of P that are chain-maximal. Felsner, Gustedt and Morvan [Felsner et al. 1999] present results on bijections between families of chains in lattices associated with P and families of interval orders defined in the ground set E.

The generation of combinatorial objects such as linear extensions and ideals of a given partially ordered set, has applications in several areas, such as combinatorial optimization and operational research. A survey on combinatorial Gray codes — i.e. methods for generating combinatorial objects in which successive objects differ in a small, pre-specified way — is presented in [Savage 1997]. Pruesse and Ruskey presented the first algorithm for listing linear extensions of P in constant amortized time (after an $O(n^2)$ preprocessing time, where $n = |E|$, [Pruesse and Ruskey 1994]). Canfield and Williamson showed that linear extensions of P can be listed in constant time per extension (after an $O(n^2)$ preprocessing time, see [Canfield and Williamson 1995]). Algorithms for the enumeration of linear extensions are also presented in [Kalvin and Varol 1983, Knuth and Szwarcfiter 1974, Varol and Rotem 1981]. An algorithm for listing the ideals of P in a combinatorial Gray code manner is presented in [Habib et al. 2001]. Algorithms for generating the ideals of P are also presented in [Bordat 1991, Jégou et al. 1995, Steiner 1986, Squire].

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Corrêa and Szwarcfiter presented characterizations of the set of extensions of the partial order P, as well as methods for the enumeration of such extensions, which require time $O(n^2\mu)$ and $O(m + n\mu)$ (where $m = |P|$, and μ is the number of extentions of P, see [Corrêa and Szwarcfiter 2005]). These characterizations are presented in terms of pairs of ideals and filters of certain suborders of P, in terms of linear extensions and sequences of ideals, and using the concept of *passive pairs* of a partial order. For this reason, the algorithms for the enumeration of extensions of P resulting from the characterizations use procedures for the generation of linear extensions and ideals of partial orders. By [Canfield and Williamson 1995], the linear extensions of P can be generated in constant time per extension (after an $O(n^2)$ preprocessing time), while the algorithm presented in [Habib et al. 2001] can be used in the generation of ideals (in $O(\Delta(P))$ time per ideal, where $\Delta(P)$ is the maximum indegree in the covering graph of P).

Characterizations of sets of extensions of an order P were studied during the development of the thesis of Aquiles Braga de Queiroz at COPPE/UFRJ. Among such characterizations, we mention characterizations of the sets of forest, interval, semi-order and weak-order extensions of an order P.

2. Characterizations of $\varepsilon(\boldsymbol{P})$

Let $E = \{e_1, ..., e_n\}$, and $P \subseteq E \times E$ a binary relation on E. The relation P is an *order* or a *partial order* on E when it is reflexive, anti-symmetric and transitive. In this case, (E, P) is said to be a *partially ordered set*. We call E the *ground set*, which will be considered finite. We say that two elements $e_i, e_j \in E$ are *comparable* in P when either $(e_i, e_j) \in P$ or $(e_j, e_i) \in P$. Otherwise, we use $e_i \parallel e_j$ in P to denote the *incomparability* between e_i and e_j. We define $I(P) = \{(e_i, e_j) : e_i \parallel e_j \text{ in } P\}$ as the set of incomparable pairs in P. If $I(P) = \emptyset$ then P is said to be a *total* or *linear order*. If P and Q are orders in a set E and $P \subseteq Q$, then Q is called an *extension* of P. If $I(Q) = \emptyset$ then Q is said to be a *linear extension* of P. Following the terminology presented in [Corrêa and Szwarcfiter 2005], let $\varepsilon(P)$ denote the set of all extensions of the order P and $\varepsilon_1(P)$ the set of linear extensions of P.

Let $E' \subseteq E$. The order $P' = P[E']$ in E', obtained by restricting P to E', is a *suborder* of P. A *chain* is a fully ordered subset $C \subseteq E$ in P. A subset $A \subseteq E$ is an *antichain* if every pair of distinct elements of A is incomparable in P. A chain $C \subseteq E$ is *maximal* when there is no other chain $C' \subseteq E$ such that $C \subset C'$. A chain $C \subseteq E$ is *maximum* when there is no other chain $C' \subseteq E$ with more elements than C. Analogously, *maximal* and *maximum antichains* are defined. The *height* $h(P)$ of the order P in E is the number of elements in a maximum chain. The *width* $w(P)$ of the order P in E is the number of elements in a maximum antichain. We denote by **n** an order formed by a chain with n elements, and by **n+p** an order formed by the disjoint union of two chains with n and p elements.

For $e_i \in E$ we denote by $E_P^-(e_i) = \{e_j \in E | (e_j, e_i) \in P \text{ e } e_i \neq e_j\}$ the set of incoming elements of e_i. Similarly, the set of outgoing elements of e_i is denoted by $E_P^+(e_i) = \{e_j \in E | (e_i, e_j) \in P \text{ e } e_i \neq e_j\}$. Such concepts can be extended to a set $S \subseteq E$, denoting $E_P^-(S) = \cup_{e_i \in S} E_P^-(e_i)$. We define $E_P^-[e_i] = E_P^-(e_i) \cup \{e_i\}$ and $E_P^-[S] = E_P^-(S) \cup S$. Similarly, we define $E_P^+[e_i]$ and $E_P^+[S]$. An element $e_i \in E$ is *maximal* (respectively, *minimal*) if $E_P^+(e_i) = \emptyset$ (respectively, $E_P^-(e_i) = \emptyset$). We denote the set of all maximal elements in P by M_P^+, while M_P^- denotes the set of all minimal elements in P. An element e_i is called *maximum* if $(e_j, e_i) \in P$, for all $e_j \in E$. Similarly, e_i is a *minimum* element if $(e_i, e_j) \in P$ for all $e_j \in E$. An *ideal* of P is a set $E' \subseteq E$ such that if $e_i \in E'$ and $(e_j, e_i) \in P$ then $e_j \in E'$. Similarly, E' is a *filter* of P when $e_i \in E'$ and $(e_i, e_j) \in P$ imply $e_j \in E'$.

An order P in E is a *forest order* if $E_P^-(e_i)$ is a chain in P for all $e_i \in E$. An order P in E is a *weak-order* if $E_P^+(e_j) = E_P^+(e_k)$ and $E_P^-(e_j) = E_P^-(e_k)$ for every pair $e_j, e_k \in E$ of incomparable elements in P. An order P in E is an *interval order* if there is a function F which assigns to each $e_i \in E$ a closed interval $F(e_i) = [a_i, b_i]$ of the real line **R** so that $(e_i, e_j) \in P$ if and only if $b_i < a_j$ in **R**. The function F is an *interval representation* of P. An interval order P is said to be a *semi-order* if there is a positive constant d such that P has an interval representation F assigning to each $e_i \in E$ an interval $F(e_i) = [a_i, a_i + d]$ such that $(e_i, e_j) \in P$ and $e_i \neq e_j$ if and only if $a_i + d < a_j$ [Queiroz and Szwarcfiter 2005].

Corrêa and Szwarcfiter describe three characterizations of the set $\varepsilon(P)$ of all extensions of an order P. The first characterization presented in [Corrêa and Szwarcfiter 2005] establishes a one-to-one correspondence between extensions of P and pairs of ideals and filters of certain suborders of P. The second characterization of $\varepsilon(P)$ is presented in terms of linear extensions and sequences of ideals. These characterizations lead to algorithms for generating extensions of P in $O(n^2)$ time per extension. The third characterization of $\varepsilon(P)$ is based on the concept of *passive pairs* and leads to an algorithm for generating extensions of P in linear amortized time per extension.

The second characterization presented in [Corrêa and Szwarcfiter 2005] relates $\varepsilon(P)$ to sequences of ideals and linear extensions. For this characterization we consider L and L' linear extensions of P, with $L = [e_1, e_2, ..., e_n]$. A *lexicographic augmentation of P relative to (L, L')*, is a binary relation B_n, obtained by the following recurrence.

$$B_0 = \emptyset \tag{1}$$

$$B_i = B_{i-1} \cup \{(e_i, e_i)\} \cup \{(e_q, e_i) | e_q \in S_i \cup E_{B_{i-1}}^-[M_{B_{i-1}}^+ \cap E_{L'}^+(e_i)]\}, \tag{2}$$

for all $1 \leq i \leq n$, where S_i is an ideal of B_{i-1}, containing $E_P^-(e_i)$.

We observe that the set $\varepsilon(P)$ of extensions of an order P can be ordered by inclusion. In this resulting partially ordered set, we have P as the only minimal element, while the linear extensions of P correspond to its maximal elements.

3. Generating Interval and Semi-Order Extensions of an Order *P*

Based on the concept of lexicographic augmentations of P relative to linear extensions (L, L') presented in [Corrêa and Szwarcfiter 2005], a characterization of the set of interval extensions of a partial order P was presented in [Queiroz 2005]. Such characterization leads to a method for generating interval extensions of a partial order P in $O(n^2)$ time per extension [Queiroz 2005]. Analogously, a characterization of the set of semi-order extensions of a partial order P and a method for generating semi-order extensions of a partial order P in $O(n^3)$ amortized time per extension were presented in [Queiroz 2005].

Applications of the concept of lexicographic augmentations originated characterizations and methods for generating forest extensions and weak-order extensions of a partial order P in polynomial time per extension.

4. Conclusion

Jayme Luiz Szwarcfiter also supervised other three theses that we may cite on the subject of partial orders at Universidade Federal do Rio de Janeiro — the thesis of Carlos Alberto da Silva Franco "Alguns Problemas Algorítmicos em Conjuntos Parcialmente Ordenados e o Teorema de Dilworth", concluded in 1983, the thesis of Adriana Figueiredo Pimenta "Sobre o Número de Saltos em Ordenações Parciais", co-supervised by Prof. Sulamita Klein and concluded in 2010, and thesis of Fabiano de Souza Oliveira "Sobre Ordens e Grafos de Intervalo", co-supervised by Prof. Márcia Rosana Cerioli and concluded in 2011. The number of supervised theses on the subject show the ability of Prof. Jayme Luiz Szwarcfiter to find different but related themes of research, that motivated many to learn and contribute.

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