Generating bicliques of a graph

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Abstract. *This is an account of the collaboration and research with Jayme Luiz Szwarcfiter and Celina Miraglia Herrera de Figueiredo during the development of the thesis of Vânia Maria Felix Dias at COPPE, "Geração de Bicliques em Grafos", finished in 2004, and its two subsequent journal papers [Dias et al. 2005, Dias et al. 2007].*

1. Introduction

Generating all configurations that satisfy a given specification is a well-studied problem in combinatorics and in graph theory suggesting many interesting problems. Among them, generating all maximal independent sets of a given graph is one that has attracted considerable attention.

Let $G = (V, E)$ be a graph, and let $X, Y \subseteq V$. We say that $B = X \cup Y$ is a complete bipartite set when both X and Y are independent sets and every vertex of X is adjacent to every vertex of Y, i.e. B induces a complete bipartite subgraph of G. If X and Y are both not empty, then B is called proper, i.e. by definition B induces a complete bipartite subgraph containing at least one edge of G, otherwise B is called degenerate. An independent (complete, complete bipartite) set is maximal, when it is not properly contained in any such set. A maximal proper complete bipartite set of G is called a (induced) biclique of G. On the other hand, when the requirement that X and Y are independent sets of G is dropped, we have a non-induced biclique.

Bicliques have been studied in some different contexts: applications in automata and language theories, graph compression, partial orders, artificial intelligence, and biology. During the development of the thesis of Vânia Maria Felix Dias at COPPE, the generation of bicliques is considered and two generation problems are solved.

2. Generating bicliques of a graph in lexicographic order

We present in [Dias et al. 2005] an algorithm that generates all bicliques of a graph in lexicographic order, with polynomial-time delay between the output of two successive bicliques. We also show that there is no polynomial-time delay algorithm for generating all bicliques in reverse lexicographic order, unless P = NP . The methods are based on those by Johnson, Papadimitriou and Yannakakis, in the solution of these two problems for independent sets, instead of bicliques.

Given two subsets S and T of an ordered set, we say that S is lexicographically smaller than T if one of the two following conditions occurs: (i) the smallest at which S

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and T of disagree is in S, or (ii) the smallest $|S|$ elements of T coincide with those of S and $|S| < |T|$. Represent by B^* the smallest lexicographic biclique of a graph G. Johnson, Papadimitriou and Yannakakis showed that there is no polynomial-delay algorithm for generating all maximal independent sets of a given graph in reverse lexicographic order, unless P = NP. Nevertheless, they presented an algorithm that generates all maximal independent sets of a graph in lexicographic order, with only polynomial delay between the output of two successive independent sets.

We extend their results to bicliques. We first show that there is no polynomial-delay algorithm for generating all bicliques in reverse lexicographic order, unless P = NP, by establishing the NP-completeness of deciding whether a given biclique is the lexicographically largest biclique. Then, we present an $O(|V|^2)$ algorithm that generates the lexicographically smallest biclique containing a given complete bipartite set of a graph $G = (V, E)$. In particular, this algorithm can generate the smallest biclique of G.

The algorithm is then used as part of an algorithm that generates all bicliques of a graph in lexicographic order, with only polynomial-time delay between the output of two successive bicliques. In order to evaluate the complexity of the generating algorithm, it is key to observe that there are known classes of graphs where the number of bicliques is exponential. The bicliques are generated in delay time $O(|V|^3)$, and the space required by the algorithm is $O(2^{|V|})$. Similarly as for independent sets, it remains open whether the bicliques can be generated in lexicographic order, in polynomial-time delay and polynomial space. However, for bicliques the question is more basic, to describe an algorithm for generating them all, in any ordering, obeying polynomial-time delay and polynomial space.

3. Generating the non-induced bicliques

Given a graph $G = (V, E)$, we say that a subset $S \subseteq V$ is part of a biclique $X \cup Y$ of G, if $S = X$ or $S = Y$. In a subsequent work [Dias et al. 2007], we show that it is NP-complete to test whether a subset of the vertices of a graph is part of a biclique. In light of this result, it might not be obvious how to obtain an algorithm for generating all the (induced) bicliques, running in polynomial time delay and polynomial space. We propose an algorithm that generates all non-induced bicliques of a graph, with these requirements of polynomial time delay and polynomial space.

In a bipartite graph all non-induced bicliques are actually (induced) bicliques. First, we show that an algorithm for generating all bicliques of a bipartite graph can be used for generating all non-induced bicliques of a general graph. Then, we describe an algorithm that generates all bicliques of a bipartite graph $G = (V, E)$ with delay time complexity $O(|V||E|)$ and required space of $O(|V| + |E|)$.

4. Conclusion

Jayme Luiz Szwarcfiter also supervised at the time another thesis on the subject of bicliques. The thesis of Marina Esther Groshaus "Bicliques, Cliques, Neighborhoods y la Propiedad de Helly" was developed at the Universidad de Buenos Aires, and concluded in 2006. The two supervisions show the interest of Jayme in the subject and his ability to find different and rich themes of research.

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