Jayme's scientific contributions in the area of Convexity in Graphs

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Abstract. *In this paper we survey some of the contributions of Professor Jayme Luiz Szwarcfiter in the area of graph convexity, in celebration of his 80th birthday.*

1. Introduction

There are many important contributions by Professor Jayme Luiz Szwarcfiter in the area of combinatorics, graph theory, and computational complexity. Through collaborations, co-supervision of master's and doctoral degrees and, most importantly, under his supervision, the authors worked and published together with professor Jayme several results in the area of convexity in graphs.

For a graph G, a set $\mathcal{C}$ of subsets of $V(G)$ is a *convexity* in G if (i) $\emptyset, V(G) \in \mathcal{C}$ and (ii) $\mathcal{C}$ is closed under intersection. Each element of $\mathcal{C}$ is called a *convex set*. The *convex hull* $H_{\mathcal{C}}(S)$ of a subset $S \subseteq V(G)$ is the smallest convex set containing S.

With few exceptions [Lima et al. 2018], most of graph the convexities studied so far are defined by a set $\mathcal{P}$ of paths (c.f. [Thompson et al. 2020]). In this case, a subset $\mathcal{C} \in V(G)$ is convex precisely when $\mathcal{C}$ contains all the vertices belonging to the paths of $\mathcal{P}$ whose extreme vertices are also in $\mathcal{C}$.

The *geodetic convexity* considers $\mathcal{P}$ as the set of all shortest paths in G [Cáceres et al. 2006, Dourado et al. 2016b, Dourado et al. 2010a, Everett and Seidman 1985]. In the *monophonic convexity* [Costa et al. 2015, Dourado et al. 2010c, Duchet 1988] $\mathcal{P}$ is the set of all induced paths. The *triangle path convexity* [Changat and Mathew 1999, Dourado and Sampaio 2016] and P_3-*convexity* consider $\mathcal{P}$ as the set of all paths with three vertices [Centeno et al. 2013, Centeno et al. 2011a, Coelho et al. 2014a, Dourado et al. 2013d].

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The P_3^*-*convexity* considers $\mathcal{P}$ as the set of all induced paths with three vertices [Araújo et al. 2018]. When $\mathcal{C}$ is the geodetic (resp. monophonic, P_3, P_3^*) convexity, we write g (resp. m, P_3, P_3^*) instead of $\mathcal{C}$.

One can say that Jayme is a pioneer among Brazilian researchers in the area of convexity in graphs and, when restricted to complexity aspects, one of the pioneers in the world. The first publication on the theme has appeared in 2006 with a study on the geodetic convexity [Dourado et al. 2006]. Since then, Professor Jayme counts more than 30 published papers dealing with convexity in graphs, besides supervising dozens of students on the subject.

Along with his coauthors, Jayme investigated several convexity invariants, in different convexities, obtaining results of complexity, bounds, and characterizations, which will be briefly summarize in the next sections. Although not exhaustive, we hope this selection of results illustrates the immense impact of Jayme in this field.

2. Hull number

Let G be a graph and $S \subseteq V(G)$. If $H_{\mathcal{C}}(S) = V(G)$ then S is a *hull set*. The cardinality $h_{\mathcal{C}}(G)$ of the smallest hull set in G is called the *hull number* of G. Being a central concept in the area, the hull number was considered in an expressive number of Jayme's papers.

In [Dourado et al. 2009], the hull number of graph under in the geodetic convexity was studied. It was shown that deciding whether $h_g(G) \leq k$ is NP-complete. On the other hand, the authors presented polynomial-time algorithms for computing $h_g(G)$ when G is a unit interval graph, a cograph, or a split graph. In [Dourado et al. 2010a] the same parameter was considered for triangle-free graphs, where bounds were presented considering restrictions of diameter, girth, and degree. Results in graph products were considered in [Coelho et al. 2022] where it was shown that $h_g(G\overline{G})$ can be determined in polynomial time, for the complementary prism $G\overline{G}$ of any graph G.

Contributions were also made considering the P_3 convexity. Polynomial-time algorithms were presented for $h_{P_3}(G)$ on trees and, more generally, chordal graphs [Centeno et al. 2011b]. Connections between minimal and minimum hull sets were explored in [Barbosa et al. 2013], where tight bounds for the size of minimal hull sets of cubic graphs were obtained and, for certain graph classes, it was proved that every minimal hull set is also minimum. On the negative side, it was proved that the hull number is NP-hard on general graphs [Centeno et al. 2011b].

[Araújo et al. 2018] introduced the P_3^* convexity and showed that, for this convexity, determining the hull number is NP-hard on planar bipartite graphs with maximum degree four, and also on subgraphs of grids, and explored the interplay between P_3 and P_3^* convexities.

In the P_3 convexity, studies considering graph products were performed showing that, given a graph G and an integer k, determining whether $h_{P_3}(G) \leq k$ remains NP-complete for the Cartesian product of graphs [Coelho et al. 2019]. On the order hand, in the same paper, the hull number of the strong and lexicographic product of two general graphs was determined, bounds were presented for the Cartesian product, $G \square H$, of two general graphs and exact values were presented for $G \square K_n$.

3. Interval number

Given a set $S \subseteq V(G)$, the *interval* $I[S]$ of S is formed by S, together with every vertex belonging to the paths of $\mathcal{P}$ whose extreme vertices are also in S. If $I[S] = S$, then the set S is a *interval set*. The *interval number*, $in_{\mathcal{C}}(G)$, of a graph G is the cardinality of the smallest interval set, that is, the cardinality of the smallest set $S \subseteq V(G)$ such that $I[S] = V(G)$.

Regarding the interval number, Jayme also made fundamental contributions. In particular, the decision problem related to determining the interval number was proved to be NP-complete for arbitrary graphs in monophonic [Dourado et al. 2010c] and geodetic convexities [Dourado et al. 2006]; for chordal and chordal bipartite graphs in geodetic [Dourado et al. 2010b] and P_3 [Centeno et al. 2009] convexities; and for bipartite graphs in P_3^* convexity [Araújo et al. 2018].

On the other hand, bounds on $in_g(G)$ were presented for triangle-free graphs and unit interval graphs in [Dourado et al. 2010c]. In the same paper, the authors provided exact values for $in_g(G)$ when G is a cograph or a split graph. In addition, polynomial time algorithms for $in_{P_3}(G)$ are known for trees, cographs and some classes of grids [Centeno et al. 2009]; and for $in_{P_3^*}(G)$ for cographs and P_4-sparse graphs [Araújo et al. 2018].

4. Carathéodory number

The *Carathéodory number* of G is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_{\mathcal{C}}(S)$, there is a set $F \subseteq S$ with $|F| \leq c$ and $u \in H_{\mathcal{C}}(F)$.

Some general results concerning the Carathéodory number of the P_3 convexity were published in collaboration with Jayme. [Barbosa et al. 2012] contains efficient algorithms to determine the Carathéodory number of the convexity of trees and, more generally, block graphs, for path convexities. On the other hand, it has been proved that is NP-hard to determine the Carathéodory number of the P_3 convexity of bipartite graphs [Barbosa et al. 2012].

Still, considering the P_3 convexity, it was presented in [Coelho et al. 2014b] a polynomial time algorithm for the computation of the Carathéodory number of a chordal graph.

Another invariant associated with the Carathéodory number, the local Carathéodory number, was also investigated. Let G be a graph and $S \subseteq V(G)$. The *local Carathéodory number* is the smallest integer l such that for every $u \in H_G(S)$ there is a set $F \subseteq S$ with $|F| \leq l$ and $u \in H_G(F)$.

In [Dourado et al. 2013b] the (local) Carathéodory number of graph convexities was studied. In particular, it was proved that Carathéodory number and local Carathéodory number are NP-complete in geodetic convexity. On the other hand, for the case of split graphs, it was shown that the Carathéodory number is at most 3 and could be easily computed. It was also proved in [Dourado et al. 2013b] that local Carathéodory number is NP-complete in P_3-convexity.

5. Convexity number

Let G be a graph. The cardinality $cn_{\mathcal{C}}(G)$ of a maximum proper convex set S of G is the *convexity number* of G. Intuitively, it is the size of a largest "non-obvious" convex set.

In [Centeno et al. 2009], it was proved that deciding whether $cn_{P_3}(G) \geq k$ is NP-complete when G is an arbitrary graph. That result was later restricted to bipartite graphs with diameter 3, in P_3 and P_3^* convexities [Araújo et al. 2018]. Other NP-completeness results were shown considering the monophonic convexity for general graphs [Dourado et al. 2010c] and the geodetic convexity for general [Dourado et al. 2006] and bipartite graphs [Dourado et al. 2012a].

On the positive side, [Centeno et al. 2009] proved that $cn_{P_3}(G)$ can be determined in polynomial time for trees, cographs, and grids. In the scope of geodetic convexity, [Dourado et al. 2012a] investigated properties of graphs with small convexity number, for which some bounds on the parameter were derived.

Regarding the monophonic convexity, [Dourado et al. 2010c] mention that $cn_m(G)$ can be determined in polynomial time for cographs. In a related concept, the same paper reports a characterization for monophonic convex sets that leads to a polynomial time algorithm for recognizing such sets.

6. Radon number

Let G be a graph. A *Radon partition* of $R \subseteq V(G)$ is a partition of R into two disjoint sets R_1 and R_2 with $H_{\mathcal{C}}(R_1) \cap H_{\mathcal{C}}(R_2) \neq \emptyset$. The *Radon number* $rd_{\mathcal{C}}(G)$ of G is the minimum integer r such that every set of at least r vertices of G has a Radon partition.

In the first paper on the Radon number of graph convexities [Dourado et al. 2012b], it was shown a tight bound for this invariant in the P_3 convexity and the extremal graphs were characterized. On the complexity side, the problem was shown to be NP-hard on the same convexity and tractable for trees [Dourado et al. 2013c]. Its tractability was also shown for split graphs, which had their Radon-partitionable sets characterized [Dourado et al. 2012c].

The geodetic Radon number was considered first in [Dourado et al. 2013a] and later proved to be solvable efficiently in [Dourado et al. 2016a].

7. Rank

Let G be a graph. A set $S \subseteq V(G)$ is *convexly independent* if $v \notin H_{\mathcal{C}}(S \setminus \{v\})$, for every $v \in S$. The *rank* $rk_{\mathcal{C}}(G)$ of G is the cardinality of a maximum convexly independent set of G. Regarding this parameter, some contributions were made in the context of P_3, geodetic and monophonic convexities.

In the paper by [Ramos et al. 2014] it was proved that the decision problem related to $rk_m(G)$ is NP-complete. The authors also showed the NP-completeness for $rk_{P_3}(G)$ for split or bipartite graphs. In contrast, they show that, in P_3 convexity, the rank can be determined in polynomial time on trees and threshold graphs, and they present upper bounds for $rk_{P_3}(G)$ on a general graph G giving as a byproduct an upper bound also for $rd_{P_3}(G)$.

Considering the geodetic convexity, in [Kanté et al. 2017] it was proved that is NP-hard to approximate the geodetic rank of bipartite graphs by a factor of $n^{1-\epsilon}$, for every $\epsilon > 0$. On the other hand, the authors describe polynomial time algorithms for finding the rank of P_4-sparse, split and distance-hereditary graphs.

8. Other contributions related to graph convexity

We remark that the contributions of Professor Jayme were not restricted to those parameters above-mentioned. For instance, there are studies concerning the Helly number in the P_3 and P_3^* convexities [Carvalho et al. 2019b, Carvalho et al. 2019a]. Partitioning problems associated with convex partitions and related problems were also target of studies, see [Centeno et al. 2010, Artigas et al. 2010, Artigas et al. 2011, Artigas et al. 2013] and also the paper on the topic in this special issue.

9. Concluding Remarks

We have surveyed some contributions by Professor Jayme Luiz Szwarcfiter to the area of convexity in graphs. All of those mentioned publications make evident how he was important to boost advances in the area. More than that, his career is made of a profound sense of continuity, in the sense of always encouraging his students to persist towards the progress of science. Undoubtedly, it caused a positive effect, as nowadays many researchers follow his footsteps and continue his work around the globe. We are happy to be part of this crowd.

An example of the great influence of Professor Jayme in the training of human resources is the Algorithms and Graphs group of the Instituto de Informática of the Federal University of Goiás. Jayme supervised, co-supervised, collaborated on research projects and scientific articles with the vast majority of professors in this group. In other words, Jayme plays an important role in the formation of this research group.

In this celebration of his 80th birthday, it is indispensable to pay Professor Jayme Luiz Szwarcfiter's special tribute for his honorable career. The authors would like to thank Professor Jayme Luiz Szwarcfiter for his outstanding contributions, teachings, and advising. Working with Jayme has been a privilege. His ingenuity, knowledge, and support made an immeasurable impact on our careers and it is a pleasure to use this opportunity to thank for all of this positive impact he made.

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