What can weighted clique graphs tell us about graph classes?

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Dedicated with love and admiration to Jayme Szwarcfiter on his 80th birthday

Abstract. *In this paper, I will briefly survey the joint work with Jayme on weighted clique graphs [Bonomo and Szwarcfiter 2014], and how they are a tool to characterize graph classes.*

1. Cidade maravilhosa, pessoas maravilhosas

Thanks to my advisor Willy Durán, I had the luck of starting an academic collaboration with Jayme exactly 20 years ago in Rio de Janeiro. The visit was around the first Latin American Workshop on Cliques in Graphs, on the occasion of Jayme's 60th birthday. I was already a big fan of Jayme's survey on clique graphs [Szwarcfiter 2003], and we started working together, with Willy, Oscar Lin and Marina Groshaus, on clique-perfect graphs and subclasses of balanced graphs, and their behaviour under the clique graph operator [Bonomo et al. 2006a, Bonomo et al. 2006b]. Ten years later, during a visit to Jayme in Rio, drinking a coffee at the COPPE, I told him I wanted to study a weighted clique graph operator, but in which, instead of giving weight only to the edges with the size of the intersection of the corresponding cliques, we could give weight to any complete set. As a kind of magician, he opened a case of his desk, and gave me a pile of old slides (transparencies) where some graph classes were characterized by means of that kind of operator. We obtained more results together, and that is the way this work was born.

If I have to mention all the good things I think about Jayme, and all the good memories I have with him, the space would not be enough. So I just want to mention that working with him is one of the most pleasant experiences of my academic life, and that Jayme is an excellent person, who stays close in the good moments, and closer in the bad ones.

2. Weighted clique graphs

The clique graph $K(G)$ of a graph G is the intersection graph of the maximal cliques of G. We consider here the generalization where weights are assigned to complete subsets of vertices of the clique graph, having certain prescribed sizes. The weights equal the cardinalities of the intersections of the cliques of G which correspond to the complete sets under consideration. This concept was considered before for weights assigned only to the edges of the clique graph, for example in [Gavril 1987], and in other works mentioned in our paper. In [McKee 2013], weights assigned to edges and triangles are considered with the aim of characterizing two graph classes, in the same spirit of our paper.

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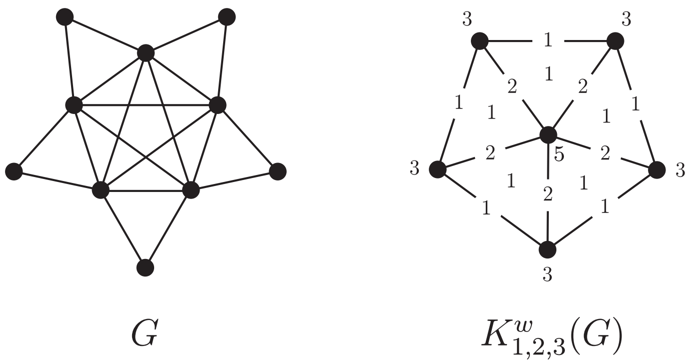

Figure 1. A graph G and its {1,2,3}-weighted clique graph.

Given a graph G and integers $m_1, \ldots, m_\ell$, the $\{m_1, \ldots, m_\ell\}$-*weighted clique graph* of G, denoted by $K^w_{m_1,\ldots,m_\ell}(G)$, is the clique graph $K(G)$ together with a function w that assigns, for each $s \in \{m_1, \ldots, m_\ell\}$ and each complete set $\{M_1, \ldots, M_s\}$ of $K(G)$, the weight $w(M_1, \ldots, M_s) = |M_1 \cap \ldots \cap M_s|$ in G. In Figure 1, a graph G and its weighted clique graph $K^w_{1,2,3}(G)$ are shown. If $\ell = 1$, we write simply m_1-*weighted clique graph*, and if $\{m_1, \ldots, m_\ell\} = \{1, \ldots, \omega(K(G))\}$, we call it *fully-weighted clique graph* of G, and denote it as $K^w_{\textit{full}}(G)$.

Clique graphs were characterized in [Roberts and Spencer 1971]. A characterization for 2-weighted clique graphs, formulated in similar terms, was presented independently in [McKee 1991] and [Roberts and Spencer 1971]. We have extended this characterization to $\{m_1, \ldots, m_\ell\}$-weighted clique graphs.

We have also looked for results of the form: *G has property X if and only its weighted clique graph has property Y*. For some graph classes, these kind of results arise naturally from their definition. This is the case for clique-Helly graphs and their generalization (p, q, r)-*clique-Helly graphs* [Dourado et al. 2009]. Diamond-free graphs and linear domino graphs have also a natural characterization in terms of their 2-weighted clique graphs, and they cannot be characterized solely by their 1-weighted clique graph, since the diamond and two triangles sharing a vertex have the same 1-weighted clique graph. In [McKee 2013], strongly chordal graphs and trivially perfect graphs were characterized by means of their $\{2, 3\}$-weighted clique graphs. It turned out that many standard graph classes admit such kind of characterization.

We proved a characterization of hereditary clique-Helly graphs in terms of K^w_3 and showed that $K^w_{1,2}$ is not sufficient to characterize neither hereditary clique-Helly graphs nor clique-Helly graphs. For chordal graphs and their subclass UV graphs, we obtained a characterization by means of $K^w_{2,3}$. We showed furthermore that $K^w_{1,2}$ is not sufficient to characterize UV graphs. We described also a characterization of interval graphs in terms of $K^w_{2,3}$ and of proper interval graphs in terms of $K^w_{1,2}$. In addition, we proved that $\{K^w_1, K^w_2\}$ is not sufficient to characterize proper interval graphs. For split graphs and line graphs, we gave a characterization by means of $K^w_{1,2}$, and also proved that $\{K^w_1, K^w_2\}$ is neither sufficient to characterize split graphs nor line graphs. Finally, we characterized trees in terms of K^w_1 and block graphs in terms of K^w_2, and showed that the latter cannot be characterized by means of their 1-weighted clique graphs.

3. Questions that are still open

We have closed the paper with some open problems, which are still open, as far as I know.

1. The first one concerns the recognition of clique graphs, which is NP-complete [Alcón et al. 2009]. The question is whether the additional information carried by weighted clique graphs with respect to clique graphs can be used to recognize them, in polynomial time in the size of $K(G)$. More precisely, can we find some set of integers $\{m_1, \ldots, m_\ell\}$ and some polynomial-time algorithm to recognize $\{m_1, \ldots, m_\ell\}$-weighted clique graphs? (Reducing from clique graphs recognition, we proved that recognition of 1-weighted clique graphs is NP-complete.)
2. In case of an affirmative answer of the above question, what would be the minimum m_ℓ and minimum size of $|\{m_1, \ldots, m_\ell\}|$ to obtain such an algorithm?
3. What is the complexity of recognizing $\{m_1, \ldots, m_\ell\}$-weighted clique graphs for different sets $\{m_1, \ldots, m_\ell\}$?
4. Is there a characterization of circular-arc graphs, in terms of their weighted clique graphs?

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