Geodesic Convexity of graphs: partition and geodetic sets*

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Abstract. *This work describes the contribution of Artigas, Dantas, Dourado, and Szwarcfiter on the topic of convexity in the context of geodesic convex partitions, covers and geodetic sets from 2008 to 2013.*

1. Introduction

Let $G = (V, E)$ be a finite, simple, and connected graph. A family $\mathcal{C}$ of subsets of V is a *graph convexity*, or convexity, if it satisfies two properties: (i) $V, \emptyset$ belongs to $\mathcal{C}$; (ii) $\mathcal{C}$ is closed under intersections. The elements of $\mathcal{C}$ are called *convex sets*.

Convexities on graphs are commonly defined using interval functions. Jayme Szwarcfiter has done relevant contributions to different classes of convexities like *monophonic convexity*, where a set S is convex if it contains all vertices lying on induced paths between any pair of vertices of S, and P_3 *convexity*, where a set S is convex if it contains all vertices lying on any P_3 between any pair of vertices of S. Our work with Jayme was focused on another class of convexity, the well-know *geodesic convexity*, defined by shortest path interval function. Therefore, every definition and result presented in this paper is related to geodesic convexity.

A *geodesic* between v and w in G is a shortest path between v and w in the graph. The *closed interval* $I[v, w]$ is the set of all vertices lying on a geodesic between v and w. Given a set S, $I[S] = \bigcup_{u,v \in S} I[u, v]$. If $I[S] = S$, then S is a *convex set*. If $I[S] = V$, then S is *geodetic*. A graph $G = (V, E)$ is called *p-convex* if there exists a partition of V into p convex sets.

The first problem that we considered was the problem of partitioning a graph into convex sets. That is, given a graph $G = (V, E)$ and an integer p, is G p-convex? We also considered a generalization of the convex partition called a convex cover. We say that a graph $G = (V, E)$ has a *convex p-cover* if V could be covered by p convex sets, i.e., there exists $\mathcal{V} = (V_1, \ldots, V_p)$, $p \in \mathbb{N}$, such that $V = \bigcup_{1 \le i \le p} V_i$; set V_i is convex and $V_i \not\subseteq \bigcup_{\substack{1 \le j \le p \\ i \ne j}} V_j$, for $1 \le i \le p$. In particular, if all sets of $\mathcal{V}$ are disjoint, $\mathcal{V}$ is a *convex p-partition* of V.

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Secondly, we considered a special case of the problem of deciding, for a graph $G = (V, E)$, whether a set $S \subseteq V$ is geodetic. The problem of determining the minimum geodetic set of a graph is NP-hard. For some sets S, formed by vertices with particular eccentricity properties, we checked for which graph class is S a geodetic set. We investigated some of these sets, especially, the *contour set* $Ct(G)$ of G defined as $Ct(G) = \{v \in V | ecc(u) \leq ecc(v), \forall u \in N(v)\}$, where $ecc(v)$ is the eccentricity of a vertex v and $N(v)$ is the set of neighbors of a vertex v.

In the next sections, we summarize the main results of our published papers.

2. Convex partition and cover

The first problem that we considered in our joint work with Jayme was the problem of partitioning a graph into convex sets. Our main contributions to this problem were presented in the paper "Partitioning a graph into convex sets" [2], published in Discrete Mathematics.

In [2] we defined the concept of a convex partition of graphs. We proved that it is NP-complete to decide whether a graph G is p-convex for a fixed integer $p \geq 2$. We showed that every connected chordal graph is p-convex, for $1 \leq p \leq n$. We also established conditions on n and k to decide if the k-th power of a cycle C_n is p-convex. Finally, we developed a linear-time algorithm to decide if a cograph is p-convex. Some preliminary results of this work were presented in [6].

For the convex cover, a generalization of convex partition, we proved that is NP-complete to decide if a general graph G has a convex p-cover for a fixed integer $p \geq 3$. We showed that all connected chordal graphs have a convex p-cover, for any $1 \leq p \leq n$. We also established conditions on n and k to decide if a power of cycle has a convex p-cover. Finally, we developed an algorithm for disconnected graphs. These results were published in the journal "Matemática Contemporânea" [4].

3. Geodetic sets

The second question considered by the authors was the problem of deciding, on a graph $G = (V, E)$, whether a set $S \subseteq V$ is geodetic. We considered this problem for some sets S formed by vertices with particular eccentricity properties, especially, the *contour set* $Ct(G)$ of G defined as $Ct(G) = \{v \in V | ecc(u) \leq ecc(v), \forall u \in N(v)\}$, where $ecc(v)$ is the eccentricity of a vertex v and $N(v)$ is the set of neighbors of a vertex v.

Our main contribution to this subject was the paper "On the Contour of Graphs" [1] published in Discrete Applied Mathematics. This was a joint work with Seiichi Yamaguchi from Shonan Institute of Technology, Japan. In [1] we established a relation between the diameter and the geodeticity of the contour of a graph. We proved that the contour is geodetic for graphs with diameter $k \leq 4$. Furthermore, for every $k > 4$, there is a graph with diameter k whose contour is not geodetic. We showed that the contour is geodetic for bipartite graphs with diameter $k \leq 7$, and for any $k > 7$ there is a bipartite graph with diameter k and non-geodetic contour. By applying these results, we improved the state of the art of the problem by showing whether the contour of cochordal, parity, and bipartite graphs are geodetic.

We also considered the *periphery* $Per(G)$ of G defined as $Per(G) = \{v \in V | ecc(v) = diam(G)\}$, where $diam(G)$ is the diameter of G. We have investigated whether $Per(G)$ is geodetic, and whether $Ct(G)$ and $Per(G)$ are monophonic sets. For a graph $G = (V, E)$, a set $S \subseteq V$ is called *monophonic* if for every vertex v of $V \backslash S$ there exists a pair of vertices $x, y \in S$ such that v belongs to an induced path between x and y.

In [3] we established a relation between the diameter and the geodeticity of the periphery of a graph. We showed that the periphery is geodetic for graphs with diameter $k = 2$ and it is not necessarily geodetic for $k \geq 3$. For $k = 3$, we characterized the graphs whose periphery is not geodetic. Similar results do not extend to graphs with diameter 4. These results lead us to solve the problem for classes of graphs like cographs, chordal, split, and threshold graphs. We also considered the monophonic convexity and described similar results as those for the geodesic convexity. We extended these results in [5], where we proved that if G is a power of a path, then $Per(G)$ is a geodetic set, and showed an infinite family of unit interval graphs for which the periphery is not geodetic. Finally, we described graphs for which the contour is geodetic and the periphery is not a geodetic set.

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