# Solving Jayme's favorite problem through a triangle catwalk

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***Abstract.*** *This is an account of our pursuit to solve the recognition of clique graphs, a problem open for 40 years, and one of Jayme's favorite problems.*

## 1. A problem proposed in 1971

A *complete set* of a graph $H = (V, E)$ is a subset of $V$ inducing a complete subgraph. A *clique* is a maximal complete set. The *clique family* of $H$ is denoted by $\mathcal{C}(H)$. The *clique graph* of $H$ is the intersection graph of $\mathcal{C}(H)$. The *clique operator* assigns to each graph $H$ its clique graph, which is denoted by $K(H)$. We say that a graph $G$ is *a clique graph* if $G$ belongs to the image of the clique operator, i.e., if there exists a graph $H$ such that $G = K(H)$. Note that the number of maximal complete sets may be exponential on the number $n$ of vertices. Consider for instance the graph consisting of an induced matching. It contains $2^{n/2}$ maximal stable sets, thus its complement is a graph with $2^{n/2}$ maximal complete sets.

Consider the time complexity of the problem of recognizing clique graphs, this is the time complexity of the following decision problem:

CLIQUE GRAPH
Instance: Graph $G$.
Question: Is there a graph $H$ such that graph $G$ is the intersection graph of the cliques of graph $H$ ?

The Helly property has been much studied with the goal of classifying the CLIQUE GRAPH problem. A family of sets $\mathcal{F} = (F_i)_{i \in I}$ is *pairwise intersecting* if the intersection of any two members is not the empty set. The *total intersection* of $\mathcal{F}$ is the set $\bigcap \mathcal{F} = \bigcap_{i \in I} F_i$. The family $\mathcal{F}$ has the *Helly property*, if any pairwise intersecting subfamily has nonempty total intersection. The complete set $C$ *covers* the edge $uv$ when $u$ and $v$ belong to set $C$. A *complete set edge-cover* of a graph $G$ is a family of complete sets of $G$ covering all edges of $G$.

Roberts and Spencer [Roberts and Spencer 1971] proved the following characterization: $G$ is a clique graph if and only if $G$ admits an edge-cover by complete sets satisfying the Helly property. The characterization by such a special edge-cover by complete sets — known as the RS-family — leads to a proof that CLIQUE GRAPH is in NP. An RS-family of size at most $|E(G)|$ gives the desired graph $H$ such that $|V(H)| \leq |V(G)| + |E(G)|$.

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Notice that for any graph $G$ the clique family $\mathcal{C}(G)$ is a complete set edge-cover of $G$, but, in general, this family does not satisfy the Helly property. Graphs such that $\mathcal{C}(G)$ satisfies the Helly property are called *clique-Helly* graphs. The characterization by Roberts and Spencer [Roberts and Spencer 1971] implies that every clique-Helly graph is a clique graph. The converse is not true: there exist clique graphs which are not clique-Helly graphs.

Actually, clique-Helly graphs were defined by Hamelink [Hamelink 1968] when he proved that every clique-Helly graph is a clique graph. The concept of RS-family generalized the proof of Hamelink from an edge-cover by cliques to an edge-cover by complete sets. Szwarcfiter [Szwarcfiter 1997] exhibited subsequently a polynomial-time algorithm to recognize clique-Helly graphs. It is remarkable that to test whether the clique family of a given graph is mutually intersecting is coNP-complete, whereas to test whether the clique family satisfies the Helly property is polynomial.

We learned about clique graphs and the challenge to establish the time complexity of the associated recognition problem by attending a course by Jayme given at the first LAGOS, at the time called GRACO, held in Fortaleza in 2001 [Szwarcfiter 2003].

## 2. The 100th anniversary of the flight of the 14 Bis

The NP-completeness of the CLIQUE GRAPH problem was settled and first presented by the authors at WG 2006, dedicated to Alberto Santos Dumont, aviation pioneer, on the 100th anniversary of the flight of his 14 Bis in Paris in October 1906. The full paper was subsequently published [Alcón et al. 2009].

The suitable NP-complete problem that provided the reduction was the following variation of 3SAT:

3SAT$_{\overline{3}}$
Instance: $I = (U, C)$, where $U = \{u_i : 1 \leq i \leq n\}$ is a set of boolean variables, and $C = \{c_j : 1 \leq j \leq m\}$ a set of clauses over $U$ such that each clause has two or three variables, each variable occurs at most three times in $C$.
Question: Is there a truth assignment for $U$ such that each clause in $C$ has at least one true literal?

The key property of the variable gadget was called the triangle catwalk. Every RS-family of the constructed graph $G_I$ must contain either the catwalk composed of false triangles, or the catwalk composed of true triangles. See Figure 1.

The NP-completeness of CLIQUE GRAPH further suggested the study of the problem restricted to classes of graphs not properly contained in the class of clique graphs. The NP-completeness of the CLIQUE GRAPH problem restricted to split graphs was first presented by the authors at WG 2011. The full paper was subsequently published [Alcón et al. 2013].

The corresponding maximization problems of finding a maximum induced subgraph that is, respectively, clique or clique-Helly were shown to be Max SNP-hard, even for maximum degree 6 graphs, and first presented by the authors at LAGOS 2007. The full paper was subsequently published [Alcón et al. 2010].

The NP-completeness of the CLIQUE GRAPH problem restricted to planar graphs is a challenging open problem.

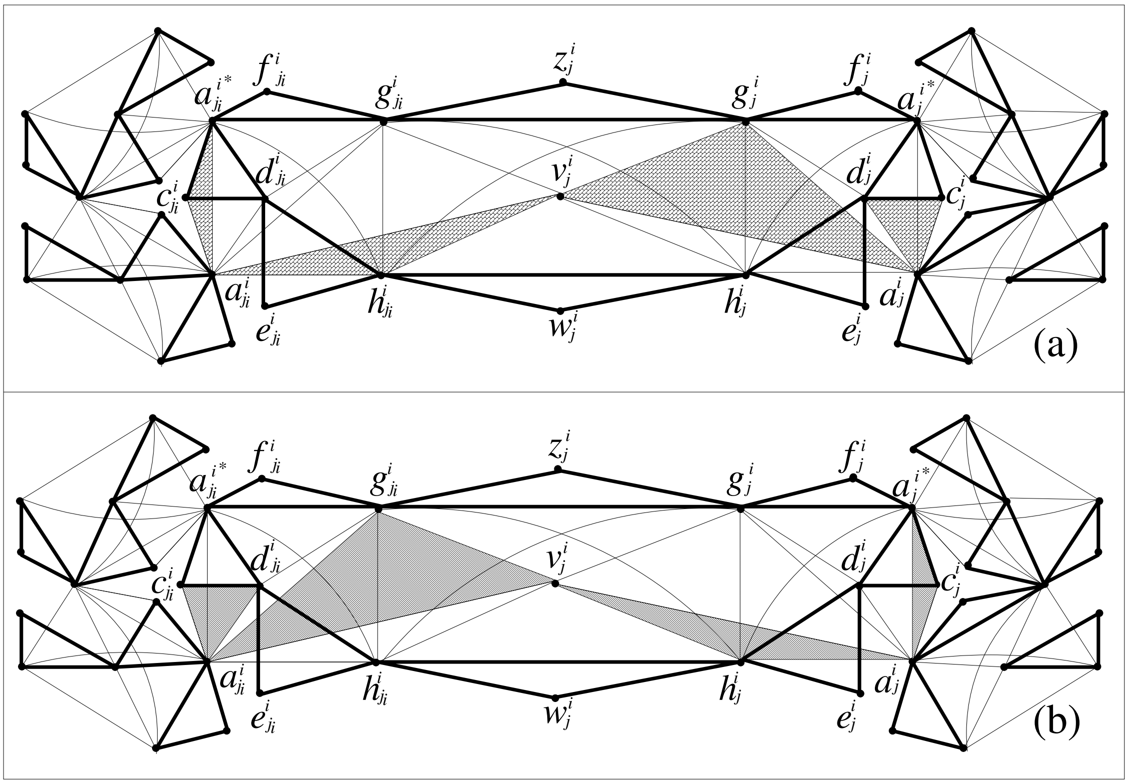


**Figure 1. The variable gadget is a triangle catwalk.**

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